Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Fall 2021

Instructions: Do **8 problems in total**, with exactly **two problems from Part I**, and **at least two problems from each of Parts II and III**. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Prove or disprove:

   (a) Compact is a homotopy invariant.

   (b) Connected is a homotopy invariant.

2. Define two continuous maps $f, g\colon X \to X$ to be topologically equivalent if and only if there exists a homeomorphism $h\colon X \to X$ with $g = hfh^{-1}$. Prove or disprove:

   (a) the maps $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ and $g(x) = 8x$ are topologically equivalent.

   (b) the maps $f, g\colon S^1 \to S^1$ defined by $f(e^{i\theta}) = e^{2i\theta}$ and $g(e^{i\theta}) = e^{8i\theta}$ are topologically equivalent.

3. Let X and Y be topological spaces and let $S = \mathrm{Top}(X, Y)$ be the set of continuous maps from X to Y. Any function $f : Z \to S$ has an adjoint function $F : X \times Z \to Y$ defined by $F(x, z) = f(z)(x)$. Call a topology on S *conjoining* if it has the property that $f : Z \to S$ is continuous whenever its adjoint $F : X \times Z \to Y$ is continuous.

   Prove that a topology on S is conjoining if and only if the evaluation map $ev : X \times S \to Y$ is continuous. Here the evaluation map is defined by $(x, f) \overset{ev}{\mapsto} f(x)$.

4. Suppose X and Y are noncompact, locally compact Hausdorff spaces. Prove that $(X \times Y)^* \cong X^* \wedge Y^*$. Here, X^* is the one point compactification of X and $\wedge$ means smash product of pointed spaces, with the base point being the point added in the compactification.

5. Show that if X is a separable metric space, i.e. it has a countable dense subset, then the topology on X is second countable.

6. Let X and Y be spaces and consider Y^X with the product topology. Prove that $A \subseteq Y^X$ has compact closure if and only if for every $x \in A$, the set $A_x := \{f(x) \colon f \in A\}$ has compact closure in Y.

Part II

7. Is the functor $\pi_1 \colon \mathsf{Top}_* \to \mathsf{Grp}$ representable?

8. Let X be the wedge sum of $\mathbb{R}P^2$ and a circle. Find all 3-fold covers of X.

9. A simple closed curve in the torus is essential if its homotopy class is non-zero. Let X be the space formed by identifying an essential simple closed curve on the torus $S^1 \times S^1$ with the boundary of a Möbius band. Use van Kampen's theorem to write down a presentation for $\pi_1 X$.

10. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the nonzero complex numbers, fix $\lambda \in \mathbb{R}_{>0}, \lambda \neq 1$ and let $\tau\colon \mathbb{C}^* \to \mathbb{C}^*$ be given by $\tau(z) = \lambda z$. Let X be the quotient of $\mathbb{C}^*$ by powers of τ, i.e. $z \sim w$ if there is an $n \in \mathbb{Z}$ such that $z = \tau^n(w)$.

    (a) What is X?

    (b) Is the quotient map $p\colon \mathbb{C}^* \to X$ a covering map? Explain why or why not.

11. In $\mathbb{R}^3$ let A denote the unit circle in the xy-plane centered at the origin, B denote the unit circle in the xy-plane centered at $(4, 0, 0)$, and C denote the z-axis. Using the fundamental group prove that the spaces $\mathbb{R}^3 - (A \cup C)$ and $\mathbb{R}^3 - (B \cup C)$ are not homotopic. (Hint: Deform the spaces.)

12. Is $\mathbb{R}P^3$ homeomorphic to the product $M_1 \times M_2$ of manifolds of lower non-zero dimensions? Explain.

Part III

13. Let X be the space formed by taking the unit sphere in $\mathbb{R}^3$ union the portion of the xy-plane lying inside the unit ball. Write down an explicit cell structure on X (*not* on a space homotopy equivalent to X) and use the cell structure to compute the homology groups of X (not via homotopy or wedge sums).

14. Let X be the space obtained by gluing three discs together along their boundary (by homeomorphisms of their boundaries). Use the Mayer-Vietoris theorem to calculate the homology groups of X.

15. Write down an explicit Δ-complex structure on the Klein bottle, and use it to compute the cup product structure on cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

16. Let $X = S^2 \vee S^4$. Show that X has the same integral homology groups as $\mathbb{C}P^2$, but is not homotopy equivalent to $\mathbb{C}P^2$.

17. Let M be a closed, connected, orientable n-dimensional manifold and let $f : S^n \to M$ be a map such that $\deg(f) \neq 0$. Compute $H_*(M; \mathbb{Q})$.