Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which “well-known” theorems you cite.

Part I

1. (a) Let \( X \) be a Hausdorff topological space. If \( \{x_n\} \) is a convergent sequence in \( X \), prove that \( \lim_{n \to \infty} x_n \) is unique.

(b) Suppose \( f : X \to Y \) is a continuous surjective function. Show that if \( X \) is compact and \( Y \) is Hausdorff, then \( f \) is a quotient map.

2. Let \( \mathbb{R}^\omega \) be the product of \( \mathbb{N} \) copies of \( \mathbb{R} \).

(a) Explain why the box and the product topologies on \( \mathbb{R}^\omega \) are not homeomorphic.

(b) Why is \( f : \mathbb{R} \to \mathbb{R}^\omega \) given by \( f(x) = (x, x, x, \ldots) \) continuous in one topology, but not in the other? Hint: Consider \( Y = \prod_{n \geq 1} \left( -\frac{1}{n} \frac{1}{n} \right) \).

3. (a) Let \( A \) and \( B \) be proper, non-empty subsets of topological spaces \( X \) and \( Y \), respectively. Show that if \( X \) and \( Y \) are both connected, then \( (X \times Y) - (A \times B) \) is connected.

(b) Let \( A \) be a non-empty subset of the metric space \( X \) and define \( f(x) = \inf \{ d(x, a) \mid a \in A \} \). Show that \( f(x) = 0 \) if and only if \( x \in \overline{A} \).

4. Let \( \{O_i\} \) be a collection of open sets which cover \( \mathbb{R}^n \). Prove that there exists a collection of open sets \( \{U_i\} \) which cover \( \mathbb{R}^n \) with the properties that for each \( i \) we have \( U_i \subseteq O_i \) and each compact subset of \( \mathbb{R}^n \) is disjoint from all but finitely many of the \( U_i \).

Part II

5. (a) Prove that a \( 2n \)-sided polygon with sides identified in pairs is homeomorphic to a closed surface.

(b) Classify the surface obtained by identifying opposite sides of a 10-gon, identified with opposite orientation as we travel in a counterclockwise direction around the 10-gon. See figure.

6. On the torus \( T = S^1 \times S^1 \), let \( \gamma = \{0\} \times S^1 \) be a meridian on \( T \). Let \( \beta \) be the equator \( S^2 \). Let \( X = T \cup S^2 \) be the space obtained by identifying \( \gamma \) with \( \beta \). Compute \( \pi_1(X, x_0) \).

7. Describe three connected non-homeomorphic 2–fold covering spaces of \( \mathbb{R}P^2 \lor S^1 \).

(a) Justify algebraically.

(b) Sketch the covers.

8. Let \( T \) be the 2–torus. Use covering spaces to prove that any map \( f : \mathbb{R}P^2 \to T \) is null-homotopic.
Part III

9. Let $F_7 = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ be the surface of non-orientable genus 7.

(a) Compute the homology of $F_7$ using cellular homology. Hint: Identify edges on a 14–gon.

(b) Compute the homology of $F_7$ using the Mayer-Vietoris sequence.

10. Let $T_1$ and $T_2$ be solid tori $D^2 \times S^1$. Let $X = T_1 \cup f T_2$ be the 3-manifold given by the attaching map $f : \partial D^2 \times S^1 \to \partial D^2 \times S^1$, with $f(x, y) = (y, x)$, i.e., attach the solid tori by identifying each meridian of $T_1$ with a longitude of $T_2$.

Use the Mayer-Vietoris sequence to compute the homology of $X$.

11. Let $M$ be a closed, connected, orientable 4–manifold with $\pi_1(M) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3$ and $\chi(M) = 6$.

(a) Compute $H_i(M, \mathbb{Z})$ for all $i$.

(b) Compute $H_i(M, \mathbb{Z}_2)$ for all $i$.

12. Let $m, n \geq 1$.

(a) Describe the cohomology rings $H^*(\mathbb{R}P^m \vee \mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}/2)$.

(b) Show that $\mathbb{R}P^m \vee \mathbb{R}P^n$ cannot be a retract of $\mathbb{R}P^m \times \mathbb{R}P^n$.

13. Let $M$ be a connected $n$–manifold. Let $D$ be an embedded closed $n$–disc in $M$. Show that if $\partial D \hookrightarrow M \setminus D^o$ is null-homotopic, then $M$ is orientable.