Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Spring 2018

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. (a) Let X be a Hausdorff topological space. If $\{x_n\}$ is a convergent sequence in X, prove that $\lim_{n\to\infty} x_n$ is unique.

   (b) Suppose $f\colon X \to Y$ is a continuous surjective function. Show that if X is compact and Y is Hausdorff, then f is a quotient map.

2. Let $\mathbb{R}^\omega$ be the product of $\mathbb{N}$ copies of $\mathbb{R}$.

   (a) Explain why the box and the product topologies on $\mathbb{R}^\omega$ are not homeomorphic.

   (b) Why is $f\colon \mathbb{R} \to \mathbb{R}^\omega$ given by $f(x) = (x, x, x, \ldots)$ continuous in one topology, but not in the other? Hint: Consider $Y = \prod_{n\geq 1}(-\frac{1}{n}, \frac{1}{n})$.

3. (a) Let A and B be proper, non-empty subsets of topological spaces X and Y, respectively. Show that if X and Y are both connected, then $(X \times Y) - (A \times B)$ is connected.

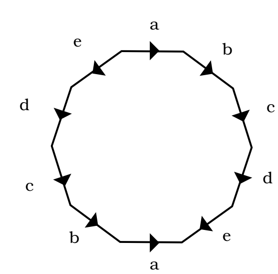
   (b) Let A be a non-empty subset of the metric space X and define $f(x) = \inf\{d(x, a) \mid a \in A\}$. Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

4. Let $\{O_i\}$ be a collection of open sets which cover $\mathbb{R}^n$. Prove that there exists a collection of open sets $\{U_i\}$ which cover $\mathbb{R}^n$ with the properties that for each i we have $U_i \subseteq O_i$ and each compact subset of $\mathbb{R}^n$ is disjoint from all but finitely many of the U_i.

Part II

5. (a) Prove that a $2n$-sided polygon with sides identified in pairs is homeomorphic to a closed surface.

   (b) Classify the surface obtained by identifying opposite sides of a 10-gon, identified with opposite orientation as we travel in a counterclockwise direction around the 10-gon. See figure.



6. On the torus $T = S^1 \times S^1$, let $\gamma = \{0\} \times S^1$ be a meridian on T. Let β be the equator S^2. Let $X = T \cup S^2$ be the space obtained by identifying γ with β. Compute $\pi_1(X, x_0)$.

7. Describe three connected non-homeomorphic 2–fold covering spaces of $\mathbb{R}P^2 \vee S^1$.

   (a) Justify algebraically.

   (b) Sketch the covers.

8. Let T be the 2–torus. Use covering spaces to prove that any map $f\colon \mathbb{R}P^2 \to T$ is null-homotopic.

Part III

9. Let $F_7 = \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{7\ \mathbb{R}P^2\text{'s}}$ be the surface of non-orientable genus 7.

   (a) Compute the homology of F_7 using cellular homology. Hint: Identify edges on a 14–gon.

   (b) Compute the homology of F_7 using the Mayer-Vietoris sequence.

10. Let T_1 and T_2 be solid tori $D^2 \times S^1$. Let $X = T_1 \cup_f T_2$ be the 3-manifold given by the attaching map

$$f\colon \partial D^2 \times S^1 \to \partial D^2 \times S^1, \text{ with } f(x, y) = (y, x),$$

i.e., attach the solid tori by identifying each meridian of T_1 with a longitude of T_2.
Use the Mayer-Vietoris sequence to compute the homology of X.

11. Let M be a closed, connected, orientable 4–manifold with $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ and $\chi(M) = 6$.

   (a) Compute $H_i(M, \mathbb{Z})$ for all i.

   (b) Compute $H_i(M, \mathbb{Z}_2)$ for all i.

12. Let $m,\ n \geq 1$.

   (a) Describe the cohomology rings $H^*(\mathbb{R}P^m \vee \mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}/2)$.

   (b) Show that $\mathbb{R}P^m \vee \mathbb{R}P^n$ cannot be a retract of $\mathbb{R}P^m \times \mathbb{R}P^n$.

13. Let M be a connected n–manifold. Let D be an embedded closed n–disc in M. Show that if $\partial D \hookrightarrow M \setminus D^\circ$ is null-homotopic, then M is orientable.