Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Spring 2021

Instructions: Do **8 problems in total**, with exactly **two problems from Part I**, and **at least two problems from each of Parts II and III**. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Prove that the Stone–Čech compactification of a countable discrete space is uncountable.

2. Prove or disprove: The property of being Hausdorff for a topological space is a homotopy invariant.

3. (a) Define the product topology on $[0,1]^{[0,1]}$.

   (b) Is the evaluation map $[0,1] \times [0,1]^{[0,1]} \to [0,1]$ continuous when $[0,1]^{[0,1]}$ is given the product topology?

4. (a) Prove that every countable metric space M containing at least two points is disconnected.

   (b) Prove that there exist countable topological spaces with more than two elements which are connected.

5. Suppose X and Y are noncompact, locally compact Hausdorff spaces. Prove that $(X \times Y)^* \cong X^* \wedge Y^*$. Here, X^* is the one point compactification of X and $\wedge$ means smash product of pointed spaces, with the base point being the point added in the compactification.

6. Suppose that the following is a pullback diagram in Top:

$$\begin{array}{ccc} F & \longrightarrow & E \\ {\scriptstyle P}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \longrightarrow & B \end{array}$$

   Prove that if p is monic then P is monic.

Part II

7. Does the functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ from the category Top_* of pointed topological spaces to the cateogry Grp of groups have a left or right adjoint $F : \mathsf{Grp} \to \mathsf{Top}_*$? Explain.

8. Let X be the torus $S^1 \times S^1$ with an open disc removed. Construct all two-fold covers of X up to covering space isomorphism, and identify how many boundary components they have.

9. Let X be the space obtained by taking two Möbius bands and gluing their boundaries together by a homeomorphism. Use van Kampen's theorem to write down a presentation for $\pi_1 X$.

10. (a) Let X be a finite cell complex and let $p : Y \to X$ be a k-sheeted covering space. Prove that $\chi(Y) = k\chi(X)$.

    (b) Let M_g denote a closed orientable surface of genus g. Prove or disprove: There exists a covering map $p : M_{10} \to M_3$.

11. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the nonzero complex numbers and consider the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$. Prove that there is no continuous map $f : \mathbb{C}^* \to \mathbb{C}$ with $\exp \circ f = \mathrm{id}_{\mathbb{C}^*}$.

12. Is $\mathbb{R}P^3$ homeomorphic to the product $M_1 \times M_2$ of manifolds of lower non-zero dimensions? Explain.

Part III

13. Let X be the space formed by taking the unit sphere in $\mathbb{R}^3$ union the segment of the x-axis lying inside the unit ball. Write down an explicit cell structure on X (*not* on a space homotopy equivalent to X) and use the cell structure to compute the homology groups of X (not via homotopy or wedge sums).

14. Let X be the double of a solid torus, i.e. the space obtained by taking two copies of a solid torus and gluing their boundaries together by the identity map. Use the Mayer-Vietoris theorem to calculate the homology groups of X.

15. Let $X = S^2 \times T^2$.

    (a) Compute the homology groups $H_*(X; \mathbb{Z})$.

    (b) Compute the cohomology ring $H^*(X; \mathbb{Z})$.

    (c) Compute $\chi(X)$.

    Please state all the theorems you are using.

16. Let X be the cell complex with one vertex, one 1-cell and two 2-cells glued using the attaching maps $z \mapsto z^2$ and $z \mapsto z^5$. Compute the homology groups $H_*(X; \mathbb{Z})$. Is X homeomorphic to a closed surface?

17. Let M be a closed, connected, orientable n-dimensional manifold and let $f : S^n \to M$ be a map such that $\deg(f) \neq 0$. Compute $H_*(M; \mathbb{Q})$.