Mortality Learning and Optimal Annuitization*

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Abstract

The disparity between the theoretical high demand for annuities and the empirical low holdings of these assets, known as the annuity puzzle, is still not completely understood in economic studies of retirement finance. Given that annuities function as longevity risk insurance, this paper assesses the effect of individuals’ own mortality learning process on annuitization. I isolate this effect by building a life-cycle model in which individuals have imperfect information of their true frailty—the one that determines the survival probability each period—and therefore have to update their beliefs about it in a Bayesian manner. Using data on subjective mortality by the Health and Retirement Study to evaluate the model, the baseline result shows that the reduction in the demand for annuities can be on average 22% solely attributable to individuals having enough information about their true mortality during retirement years. In this manner I show how the level of annuitization is associated with the difficulty individuals have for learning their own true mortality. Additionally, given the emphasis on longevity risk of the model, I study the risk-aversion effect of willing to annuitize more when uncertainty about mortality increases.

1 Introduction

Understanding the motivations driving retirement savings and other retirement financial decisions are key for the proper design of correspondent pension and retirement policies. Yaari (1965) shows that in a world with complete markets, intertemporally separable utilities, and uncertain lifespans, it is optimal for risk averse individuals to hold their entire wealth in fair priced annuities due to the potential high payment at advanced ages—a feature known as mortality credit. Nevertheless, this theoretical result is not benevolent by the data because annuities holdings of retired individuals, in the United States and other developed countries, are very low. This anomaly is referred to as the annuity puzzle.1 In order to understand this phenomenon authors sought to account for other different

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1Friedman and Warshawsky (1990) is among the first studies trying to understand the annuity puzzle by reporting that only 2% of the elderly population sampled in the Retirement History Survey hold annuities. More recently, using data from from the Health and Retirement Study, Hosseini (2015) reports that only about 3% of total retirement wealth in the United States has been privately annuitized.
aspects influencing individuals’ decision of annuitizing or not (among the formulation of more realistic models), but these new approaches only proved the robustness of the complete annuitization result originally found in Yaari (1965).

In this paper I examine the role of the evolution of subjective mortality beliefs, throughout the life-cycle, on the annuity puzzle. Given that annuities function as longevity risk insurance it is important to understand the longevity expectations of individuals before and during the retirement years, especially because these longevity expectations are not necessarily accurate, as it is commonly assumed in a typical life-cycle model. If individuals perceive an stochastic distribution of their survival chances, a greater level of resolved longevity uncertainty in the latter years of life can reduce the demand for annuities during retirement: the better an individual understands her survival chances, the less appealing a longevity risk insurance is since her idiosyncratic longevity is almost perfectly foreseen, and consequently, annuity payments are not better than using wealth with discretion at advanced ages. To analyze this effect, I study the structural formation of subjective survival probabilities as a learning process of objective survival probabilities in an otherwise basic partial equilibrium life-cycle model. I assume that in order to infer their survival probability chances, individuals have to learn about their own frailty parameter which makes every individual survival distribution idiosyncratic. This model then gives a theoretical counterpart to the data on subjective mortality beliefs found in the Health and Retirement Study, which is thus used to assess the performance of the model.\(^2\)

To find a compelling theoretical reason for the lack of annuitization has demonstrated to be a difficult task. In general, previous papers have shown that the potential mortality credit available to individuals, thanks to their annuities holdings, not only can increase consumption, but can also smooth it during the life-cycle (Reichling and Smetters 2015, Peijnenburg et al. 2016). Nevertheless, incomplete annuitization, although still high, has been deemed possible by some authors. Davidoff, Brown and Diamond (2005) found optimal annuitization of as low as 75% by assuming habit formation and incomplete annuity markets; Peijnenburg et al. (2016) though show that market incompleteness in a more general environment can still yield full annuitization. On the other hand, Reichling and Smetters (2015) have found considerably lower—and theoretically plausible—levels of annuitization thanks to a mechanism that allows them to model stochastic mortality risk (as opposed to deterministic mortality risk originally assumed in Yaari’s model). Even though they emphasize the importance of modeling stochastic survival probabilities for the annuitization decision, health shocks in their model not only affect survival probabilities, but also income and medical expenditures—features that are crucial for the decision of annuitization as well. Therefore, the isolated effect of survival expectations on optimal annuitization, in a model with stochastic mortality, remains as a research question.

\(^2\)Subjective mortality has captured the attention of different fields thanks to the availability of data from the HRS: Hurd and McGarry (1995, 2002) use subjective mortality data as predictor of actual mortality; Heimer et. al. (2015) show how subjective mortality data can explain retirement savings puzzles; Gan et. al. (2015) use a life-cycle model to show how subjective mortality data is more adequate to fit this type of models; Sun and Webb (2011) study early claiming of Social Security using subjective mortality data and the implications for medical underwriting. An important difference of the present work with all these past studies is that I use the data on subjective mortality to model a structural learning process of actual mortality, giving a direct theoretical interpretation to the subjectiveness attribute of beliefs.
The contribution of this paper is both to provide a new framework of a mortality learning process capable of generating a distribution subjective survival probabilities, and the empirical evaluation of this model (using subjective survival probabilities data from the Health and Retirement Study) in determining how much resolved longevity uncertainty can account for the low levels of annuitization. Apart from being a new plausible reason for a lower theoretical level annuitization, this result is also important because it takes place in a less stringent life-cycle environment, that is, with intertemporally separable utilities and complete annuity markets. Furthermore, the stochastic nature of survival expectations is determined without the use health shocks, which avoids the potential problem of having an effect on medical expenditures at advanced ages, a situation that can distort the observed effect of survival expectations on the annuitization decision.

To obtain the empirical features of the learning process I use a Simulated Method of Moments to minimize the distance between the model’s predicted average subjective survival probability per age and its empirical counterpart found in the Health and Retirement Study. Individuals start with common priors of their individual mortality, but eventually have a better understanding of their individual mortality as they age; or in other words, they update their mortality beliefs in a Bayesian manner. This can be seen by examining how the dispersion of mortality beliefs tends to increase by age, as Figure 1 shows. At first individuals seem to share the same belief priors of their mortality (in this case reflected by their belief in their chance to survive to age 80) which results in a less dispersed population of survival beliefs. Eventually though each individual has a better understanding of her idiosyncratic mortality, and as a consequence the dispersion of survival beliefs becomes wider. This fact allows me to contrast with previous models of stochastic mortality like Reichling and Smetters (2015): the present value of the annuity is not calculated using a stochastic process of the surviving chances of the individual, but instead, is based on the projection of subjective surviving chances according to a current belief of the individual’s own frailty.

![Figure 1: Mortality Beliefs Dispersion per Age](image-url)

Source: Health and Retirement Study, 2010
The baseline result shows that, solely attributable to resolved longevity uncertainty, the demand for annuities in the first year of retirement is on average 22% smaller. Furthermore, I show how annuitization at retirement ages is associated with the impeding noise at the time of learning, and with the initial beliefs of the frailty distribution. In other words, an individual who has less difficulty in learning about her true mortality, due to either a low noise in the learning process or due to having accurate initial beliefs, will be less willing to buy annuities. On the other hand, somebody who has more difficulty in learning about her mortality would plan her wealth investment with less discretion, increasing the value of a longevity insurance offered by an annuity. Additionally, given that the mortality credit is also capable of smoothing consumption in the life-cycle, I quantify the effect of willing to annuitize more when risk-aversion increases. Lastly, I conclude that the impact of the replacement ratio of a Social Security program has a negligible impact on annuitization in the present context.

The case for subjective mortality (interpreted as the result of a learning mortality process) and annuitization is strengthened when we consider other past attempts of incorporating stochastic mortality, as explained in Brugiavini (1993). This work showed that even when a health shock reduces the present value of an annuity, individuals would still annuitize and pool this risk by annuitizing wealth early in life. In the present model, even though subjective mortality is stochastic, this is not possible because instead of expecting future shocks, individuals are refining beliefs. Consequently, early annuitization is not a solution—individuals have a belief of what their survival chances are, not an expected value of them. Furthermore, this paper is in line with past studies showing empirical evidence of cognitive constraints as one explanation for the observed low levels of annuitization. For example, Brown et. al. (2013) show evidence that individuals have great cognitive difficulty when valuing annuities and therefore recur to simple heuristics for these financial decisions.

In what follows of the paper, section 2 describes the model, section 3 describes the empirical strategy, section 4 explains the results of simulations and experiments, and section 5 concludes.

2 Model

The key feature of the model is the fact that individuals do not know their true survival probabilities when forming optimal decisions about consumption and precautionary savings. Instead, they use their own subjective survival probabilities. In this section I describe a life-cycle model in which individuals slowly learn about their true survival probabilities, which in turn are determined by their own—to be learned—true frailty. The model follows closely the one developed by Guvenen (2007) concerning learning dynamics about individuals’ characteristics.

2.1 Individuals

An individual \(i\) has a risk of mortality determined by her own actual frailty \(\delta^i\). Following Vaupel, Manton, and Stallard (1979) and Manton, Stallard, and Vaupel (1981), each frailty \(\delta^i\) determines an (objective) survival probability each period \(t\), \(P_t(\delta^i)\). Therefore, if any individual knew the true value
of her frailty $\delta^i$ she would use $P_t(\delta^i)$ to calculate the risk of mortality throughout her life-cycle. As mentioned, in this model I assume instead that each period $k$ they form a belief of their actual frailty $\hat{\delta}^i_k$. I refer to $\hat{\delta}^i_k$ as idiosyncratic subjective frailty since it will allow me to define later an idiosyncratic subjective survival probability $P_t(\hat{\delta}^i_k)$.

Consider an economy populated with these individuals who enter the labor market when born. Idiosyncratic subjective frailty $\hat{\delta}^i_k$ is formed each period and evolves in time according to each individual’s learning dynamics. They retire at period $t = t_{R E T}$, and even though they are forming beliefs about survival probabilities they have certainty that they cannot survive longer than age $T$.

For every age $t \leq T$ an individual receives a flow utility from consumption $c_t^i$, and furthermore, the forward-looking utility at period $t$ the individual believes she has is

$$U_t = \sum_{s=t}^{T} \beta^s P_s(\hat{\delta}^i_s) u(c_s^i).$$

where $u(c) = c^{1-\gamma}$. 

Suppose $\rho_t(\hat{\delta}^i_k)$ is the individual’s subjective probability of survival at the beginning of period $t$ conditional on surviving until the end of period $t - 1$, based on period $k$ subjective frailty $\hat{\delta}^i_k$. We have then $P_t(\hat{\delta}^i_k) = \prod_{i=0}^{t-1} \rho_i(\hat{\delta}^i_k)$. The certain end of life at age $T$ implies that $\rho_{T+1} = 0 \forall i$.

### 2.1.1 Formation of Subjective Survival Probabilities

Frailty $\delta^i$ determines the (unconditional) probability of survival of individual $i$ at each age $t$, $P_t(\delta^i)$. The higher the frailty, the lower this probability is for each age $t$.\(^3\)

In order to pin down the unconditional survival probability as a function of frailty $P_t(\hat{\delta}^i_k)$, assume individuals are aware of the effect frailty has on the force of mortality, that is, the instantaneous rate of mortality as a function of frailty. I assume this mechanism is identical across individuals, as proposed by Vaupel, Manton, and Stallard (1979), and Manton, Stallard, and Vaupel (1981).\(^4\) Specifically, let $h_t(\delta)$ be the force of mortality of individual $i$ at age $t$ for any $\delta^i$, such that $h_t^i(\delta^i) > 0$. Assume that for any two individuals $i$ and $j$ we have

$$\frac{h_t(\delta^i)}{h_t(\delta^j)} = \frac{\delta^i}{\delta^j}.$$

Furthermore, assuming $j$ is the standard individual such that $\delta^j = \delta^{std} = 1$, we have

$$h_t(\delta^i) = \delta^i h_t^{std},$$

where $h_t^{std}$ is the force of mortality of the standard individual whose frailty has been normalized to 1. This in turn determines the cumulative mortality hazard $H_t(\delta^i)$ for an individual with frailty $\delta^i$ as

$$H_t(\delta^i) = \int_{0}^{t} h_s(\delta^i)ds = \delta^i \int_{0}^{t} h_s ds \equiv \delta^i H_t^{std}.$$  

\(^3\)Since I am assuming individuals form their subjective probabilities of survival with current knowledge of their frailty, I also assume they are aware there is heterogeneity across individuals with the existence of a set of individual frailty types $\Delta = [\delta, \tilde{\delta}]$, and furthermore, I assume they know there is a well defined cumulative distribution $F \in \Gamma(\Delta)$.

\(^4\)This framework is also used in Hosseini (2015).
where $H_t^{std}$ is the cumulative mortality hazard of the the standard individual. $H_t(\delta^i)$ determines the unconditional survival probability $P_t(\delta^i)$ for individual $i$, which in turn then will be a function of $H_t^{std}$, since

$$P_t(\delta^i) = \exp(-H_t(\delta^i)) = \exp(-\delta^i H_t^{std})$$

This last expression allows us to define now the concept of subjective unconditional survival probabilities.

**Definition.** A subjective unconditional survival probability based on period $k$ subjective frailty $\hat{\delta}_k$ is defined as

$$P_t(\hat{\delta}_k) = \exp(-H_t(\hat{\delta}_k)) = \exp(-\hat{\delta}_k H_t^{std})$$

To make this model computable we also need to assume individuals are aware of the standard individual with frailty $\delta^{std} = 1$. This allows us to also interpret $P_t(\hat{\delta})$ as an individual’s deviation-of-the-standard belief, that is, the belief about how far her frailty deviates from the standard frailty. This is the key aspect of the definition above.

It is important to note that, in general, there is no rule about who is considered the standard individual. This will allow us to make this model tractable in the next section: we can identify the standard individual as the representative individual who has an objective survival probability identical to the aggregate survival data found in life tables.\(^5\)

### 2.1.2 How do individuals learn their own frailty?

I assume there is a noisy signal of frailty in order to model the fact that individuals do not observe it directly. I also assume individuals learn about their own frailty in a Bayesian manner through random realizations of the noisy signal. Specifically, I define $m_t$ to be the sum of the signal and the noise. As signal I use a function of frailty $d_t(\delta^i)$, and the noise is defined as $e_t^i \sim N(0, \sigma_e^2)$:

$$m_t^i = d_t(\delta^i) + e_t^i$$

This framework—plus the specification of $d_t(\delta^i)$ to be discussed below—allows the individual to fully learn about $\delta^i$ at latter periods in life (high values of $t$). Before that, even though the individual observes the realization of the noisy observation $m_t$, she still has imperfect information about her frailty $\delta^i$.

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\(^5\)I choose this survival standard because, typically, life tables have been used in the life cycle literature to infer the survival probability of a representative individual, as in Huggett (1996). It is therefore plausible to assume that all individuals are aware of this aggregate measure.
Bayesian Learning  Once the object to be learned is formulated, the dynamic Bayesian process of updating beliefs can be specified. For this purpose, this learning process can be expressed as a Kalman filtering problem using a state-space representation. Given that the state variable being learned—the unobserved frailty $\delta^i$—is a scalar, there is no need to specify a state equation. The observation equation on the other hand, or the specific way in which what is unobservable affects what is observable, corresponds to the specification of the noisy signal.

Before I describe the learning process I specify the form of $d_t^i(\delta^i)$, which regulates the speed of learning in a Kalman filtering process. I choose the form

$$d_t^i(\delta^i) = \gamma t^\gamma \delta^i$$

which uses the parameter $\gamma$ as a regulator of the speed of learning, given that this formulation is a function of a trend. As mentioned above, this formulation allows for full learning at latter stages of life. The lower the value of $\gamma$ the longer complete learning will be delayed. A plausible interpretation of this formulation is that aging reveals true frailty at the latter years of life. This is in line with the findings of Heimer et. al. (2015) where the authors find that subjective survival probabilities are based more on natural aging as individuals grow older.

With this framework now set we can formulate the Kalman update equations for the optimal (dynamic) learning of $\delta^i$, and its variance $\sigma_{\delta^i}^2$,

$$\hat{\delta}_t^i = \hat{\delta}_{t-1}^i + G_t [m_t^i - \gamma t^\gamma \hat{\delta}_{t-1}^i],$$

$$\sigma_{\delta^i,t}^2 = \sigma_{\delta^i,t-1}^2 - G_t \gamma t^\gamma \sigma_{\delta^i,t-1}^2,$$

where $G_t$ is the Kalman gain at time $t$ given by

$$G_t = \frac{\gamma t^\gamma \sigma_{\delta^i,t-1}^2}{(\gamma t^\gamma)^2 \sigma_{\delta^i,t-1}^2 + \sigma_e^2}$$

As in Guvenen (2007), we must specify the initial values $\hat{\delta}_0^i$ and $\sigma_{\delta^i,0}^2$, as these represent the information with which individuals enter the economy.

At this point it is useful to fix ideas about mortality beliefs and mortality deviations from the standard individual’s mortality. In the present model, we can identify the standard individual as the representative individual who has a survival probability identical to the aggregate survival data, for which a normalized frailty $\delta^{std} = 1$ is defined. Actual (idiosyncratic) frailty $\delta^i$ is derived by assuming a distribution around $\delta^{std} = 1$, but throughout most of the life-cycle, each $\delta^i$ is not observed by the correspondent individual. The process of learning $\delta^i$ then implies that the distribution of learned frailties $\hat{\delta}_t^i$ at time $t$ is not the same as the distribution of $\delta^i$, and consequently, the distribution of subjective mortality beliefs will not be the same as the actual mortality distribution of individuals, unless $\delta^i$ is fully learned by all of them at advanced ages.
2.2 Annuities and Bonds

In this model wealth $a_t$ can be invested in either risk-free bonds or annuities. The total portfolio return $R$ then is the sum of both types of returns. While the risk-free bond return $r$ is constant, the one-period return of the annuity will be determined by the price of the annuity at the time of buying it and at the time of selling it.

As in Reichling and Smetters (2015), to be able to compare with any other investment, I assume annuities pay a dividend of $1. Correspondingly, the actuarially fair annuity price at time $t$ according to the current frailty belief of the individual is

$$q_t(\hat{\delta}_t) = \rho_{t+1}(\hat{\delta}_t) + \frac{\prod_{i=t+1}^{t+2} \rho_i(\hat{\delta}_t)}{(1+r)^2} + \cdots + \frac{\prod_{i=t+1}^{T-t} \rho_i(\hat{\delta}_t)}{(1+r)^{T-t}}$$

which clearly shows its dependency on the subjective frailty of the individual. It is important to note that this price takes into account all future payments until certain end of life at age $T$. This is another advantage of this model given that annuities are able to reflect perpetuity payments even at advanced ages, regardless of the age at which the annuity is being priced.

The annuity one-period return then according to the current frailty belief is

$$\phi(\hat{\delta}_t) = \frac{1 + q_{t+1}(\hat{\delta}_t)}{q_t(\hat{\delta}_t)} - 1$$

which displays the key difference of this model with the one in Reichlig and Smetters (2015): the price of the annuity for the next period $q_{t+1}(\hat{\delta}_t)$ depends on current beliefs (frailty belief), rather than expectation computations.

As a consequence, the portfolio return $R$ an individual believes is facing also depends indirectly on $\hat{\delta}_t$:

$$R(\hat{\delta}_t) = \theta_t \phi(\hat{\delta}_t) + (1 - \theta_t)r$$

where $\theta_t \in [0, 1]$ is the share invested in annuities at time $t$.

2.3 Recursive Formulation

At any given period $t$, an individual receives her previously determined wealth $a_t$ and her disposable current income (net earnings before retirement and Social Security benefits when retired), and subsequently observes the realization of the noisy signal $m_t$. She forms then a belief of her frailty $\hat{\delta}_t$.

With this information, the individual decides between consumption and savings. After that decision is made, the individual decides to invest her wealth (savings) in annuities and risk-free bonds taking into account the annuity one-period return $\phi(\hat{\delta}_t)$. That is, the individual chooses the value $\theta_t \in [0, 1]$. Lastly, before the period ends, the agent sells her annuities and bonds holdings, determining this way her wealth for the next period $a_{t+1}$.

This problem can now be formulated in a recursive manner. The state vector is formed by assets $a_t^i$, the noisy signal $m_t$, and the frailty belief $\hat{\delta}_t^i$. Define $V_t^i$ as the value function of a $t$ year old individual, the dynamic problem is then
\[ V_t^i(a_t^i, \hat{\delta}_t^i, m_t) = \]
\[ \max_{c_t, a_{t+1}, \theta_t} \left\{ u(c_t^i) + \beta \rho_{t+1}(\hat{\delta}_t^i) E \left[ V_{t+1}(a_{t+1}^i, \hat{\delta}_{t+1}^i, m_{t+1}) | \hat{\delta}_t^i \right] \right\} \]

s.t.
\[ a_{t+1} = (\theta_t \phi(\hat{\delta}_t^i) + (1 - \theta_t) r) a_t + \bar{w}_t (1 - \tau) - c_t \quad \text{for } t < t_{RET} \]
\[ a_{t+1} = (\theta_t \phi(\hat{\delta}_t^i) + (1 - \theta_t) r) a_t - c_t + ss_t \quad \text{for } t \geq t_{RET} \]
\[ \hat{\delta}_t^i = \delta_{t-1}^i + G_t[m_t^i - \gamma t^i \hat{\delta}_{t-1}^i] \]
\[ \sigma_{\delta, t}^2 = \sigma_{\delta, t-1}^2 - G_t \gamma t^i \sigma_{\delta, t-1}^2 \]
\[ a_{t+1}^i \geq 0 \]

where
\[ G_t = \frac{\gamma t^i \sigma_{\delta, t-1}^2 + \sigma_{\delta, t-1}^2}{(\gamma t^i)^2 \sigma_{\delta, t-1}^2 + \sigma_{\delta, t-1}^2} \]
\[ m_t^i = \gamma t^i \delta_t^i + e_t^i \]
\[ \phi(\hat{\delta}_t^i) = \frac{1 + \theta_{t+1}(\hat{\delta}_t^i)}{q_t(\hat{\delta}_t^i) - 1} \]

and where \( \beta \) is the discount factor, \( \bar{w}_t \) is the individual's wage in the current period, \( ss_t \) is the Social Security benefit, and \( \tau \) is the Social Security tax.

Typical of this type of models, there is no analytical solution for this problem, so I compute the solution with numerical methods.

### 2.4 Social Security

Because an amount is being paid in perpetuity, the presence of a Social Security benefit is important as an annuity substitute in the model (Feldstein, 2005), even though, Caliendo et al. (2014) show that Social Security benefits are not a perfect substitute for annuity payments since agents are not optimally choosing to invest in this program. In order to control for any imperfect substitution for annuities, I include a simple form of Social Security program without redistributive roles. I adopt a model of balanced government spending such that Social Security benefits and taxes conform
\[ \sum_{j=1}^{t_{RET}-1} \bar{w}_{j, T} = \sum_{j=t_{RET}}^{T} ss_j. \]

### 3 Empirical Strategy

The empirical strategy consists in first calibrating, through a Simulated Method of Moments, the governing parameters of the frailty learning process using the HRS data on subjective mortality

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6Social Security benefits are not the result of an optimization decision between consumption and savings in the life-cycle.
beliefs, and subsequently, simulate the learning process of frailty to then compute the present value of an annuity at each age. In this manner, we can track the decision of the individual of whether to annuitize or not her wealth (partially or completely) as she ages.

3.1 Data

For the first part, I use the subjective survival chances responses of the tenth wave (year 2000) of the Health and Retirement Study. The HRS survey includes a section of expectations in which respondents are asked about different expectations regarding survivorship, work, health, etc. For the case of subjective survival probabilities, they are asked to assign a numerical value between 0 and 100 with the following explanation:

“Next I have some questions about how likely you think various events might be. When I ask a question I’d like for you to give me a number from 0 to 100, where "0" means that you think there is absolutely no chance, and "100" means that you think the event is absolutely sure to happen.”

with the specific question being:

“(What is the percent chance) that you will live to be _target age_ or more?”

These questions about survivorship have a larger sample for the target ages of 75 and 80. I restrict the lowest possible age of the respondents to 41 in the sample to use in order to have a representative sample per age. This gives us enough data to see how the expectations of survival for two target ages change throughout cohorts. The standard individual for which frailty is set to unity is taken from the life tables of Bell and Miller (2005).

Figure 2 shows the average per age of these subjective values. The distance between both averages seems consistent throughout all ages; this allows us to believe in the existence of individual subjective frailties determining the observed subjective survival probabilities, and therefore, conclude that the evolution of the subjective probabilities for a target age is not independent from that from another target age.
3.2 Simulated Method of Moments

Interpreting each belief as a conditional subjective belief, that is,

\[ \rho_{75}(\hat{\delta}_k) \text{ for } k = 40...65 \]

and

\[ \rho_{80}(\hat{\delta}_k) \text{ for } k = 40...69 \]

we have 25 first moments for \( \rho_{75}(\hat{\delta}_k) \) and 29 first moments for \( \rho_{80}(\hat{\delta}_k) \), making a total of 54 moments to match for the four parameters of the frailty learning process that we are trying to calibrate (\( \sigma^2_e \), \( \sigma^2_{\delta_i,0} \), \( \hat{\delta}_0 \), \( \gamma \)), plus the variance of the actual frailty distribution found in the data \( \sigma^2_A \). It is important to note that, in virtue of assuming that older individuals know with more accuracy their objective mortality, we are capable of obtaining the actual distribution of frailties, which is approximately equal to the distribution of frailty beliefs of the oldest individuals.

The calibrated parameters then are given by

\[ \hat{b} = \arg\min_b g(\sigma^2_e, \sigma^2_{\delta_i,0}, \hat{\delta}_0, \gamma, \sigma^2_A)'Wg(\sigma^2_e, \sigma^2_{\delta_i,0}, \hat{\delta}_0, \gamma, \sigma^2_A) \]

\[ \text{for } b = \sigma^2_e, \sigma^2_{\delta_i,0}, \hat{\delta}_0, \gamma, \sigma^2_A \]

where \( g(\sigma^2_e, \sigma^2_{\delta_i,0}, \hat{\delta}_0, \gamma, \sigma^2_A) \) is a 54 \( \times \) 1 matrix measuring the distance between the sample moments and the model moments concerning \( \rho_{75}(\hat{\delta}_k) \) and \( \rho_{80}(\hat{\delta}_k) \), and \( W \) is a 54 \( \times \) 54 weighting matrix. \footnote{For simplicity, and given that with this method we are not interested in doing inferential analysis, I match only first moments and restrict \( W \) to be an identity matrix.} Table 1 shows the calibrated values.
Table 1. SMM calibrated parameters

<table>
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<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^2_e$</td>
<td>Noise Variance</td>
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</tr>
<tr>
<td>$\sigma^2_{\delta,0}$</td>
<td>Initial Belief of Frailty Distribution Variance</td>
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<tr>
<td>$\delta_0$</td>
<td>Initial Belief of Frailty</td>
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<tr>
<td>$\gamma$</td>
<td>Speed of Learning Regulator Parameter</td>
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<tr>
<td>$\sigma^2_\lambda$</td>
<td>Actual Frailty Distribution Variance</td>
<td>60</td>
</tr>
</tbody>
</table>

Once the path of the state variables $\rho(\hat{\delta}_k)$ and $m_t$ is simulated with the calibrated parameters I proceed to solve for the dynamic programming problem of section 2.3 over them.

Income $\bar{w}$ is set to match the average replacement of the Social Security benefit $ss$ to 45%, as estimated by Hosseini (2015). Social Security tax $\tau$ then is calibrated to match the expression described in section 2.4. Table 2 summarizes the rest of the parameters being calibrated for the partial equilibrium computation.

Table 2. Baseline Parameterization

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
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<tr>
<td>$T$</td>
<td>Age of Certain Death</td>
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</tr>
<tr>
<td>$t_R$</td>
<td>Age of Retirement</td>
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<tr>
<td>$\tau$</td>
<td>Social Security Tax</td>
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<tr>
<td>$r$</td>
<td>Annual Risk-free Interest rate</td>
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<td>$\beta$</td>
<td>Time Discount Factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Coefficient Relative Risk Aversion</td>
<td>3</td>
</tr>
</tbody>
</table>

4 Simulation Results

4.1 Baseline model

For the baseline model Figure 3 shows the results. The salient feature in this simulation is the net effect of the individual learning about her mortality: the decreasing nature of optimal annuitization in aging. From almost complete annuitization at the beginning of the individual’s work life, we can see how the fraction of wealth annuitized reduces to approximately 78% solely because the individual learned about her true mortality chances, and therefore, can partially disregard the insurance that an annuity offers. In other words, the need to insure against longevity risk decreases. In this model, this is true because the more the individual knows about her true frailty, the more confident she is about her subjective survival probabilities.

Nevertheless, incentives change as the individual retires. Another important force is gaining strength in the optimal annuitization decision during this period: the mortality credit. Even though in this setup such payment would be subjective, what is important is that it still influences the decision to annuitize or not, and therefore, at advanced ages, not only a lot of the uncertainty regarding
true frailty is resolved, but also a higher annuity payment is apparent for the individual. All these counteracting forces shape the optimal annuitization profile shown in Figure 3.

4.2 Robustness

4.2.1 Noise

The choice of the variance $\sigma^2$ can increase or decrease the speed of learning, as shown in Figures 4. We can see that when $\sigma^2$ is half its baseline value the individual learns about her actual frailty quickly, and therefore, her subjective survival probabilities are pretty well aligned with her actual mortality. Realizing this, the individual does not find the need to hedge against longevity risk anymore, and drops the high level of annuitization quickly in the life-cycle. The converse thus is that if we double the noise form its baseline value, the individual annuitizes more throughout the lifecycle.
4.2.2 Initial Frailty Distribution Beliefs

The initial belief for the frailty distribution $\sigma_{\delta,0}^2$ has a large impact for the annuitization process in the life-cycle, as Figure 5 shows. Like the noise parameter in the previous section, this belief determines how fast an individual will learn about the true frailty, by establishing the initial uncertainty about the actual frailty distribution. We can see in this figure that when underestimating the actual frailty distribution at the beginning of life, at just 6% of the actual frailty variance, it is more difficult for an individual to learn about her true frailty. On the other hand, annuitization levels jump when we double the baseline percentage parameter from 12% to 24% of the actual frailty variance.
4.2.3 Replacement Ratio

The replacement ratio is an important reference to see the level of annuitization substitution a Social Security program offers. It is often assumed that annuity markets are not thick due to the presence of perpetuity payments (Caliendo et al. 2014). Therefore, it is important to see what is the effect of the replacement ratio on annuitization in the context of mortality learning. Figure 6 shows that this effect is small. Effectively, during retirement years, by lowering the replacement from 45% to 15%, annuitization decisions are higher. But overall this effect is not large. The same is true when the replacement ratio increases to 60%, the effect is as expected but overall not significant.
4.2.4 Risk Aversion

In this model we can observe that when individuals become more risk-averse, annuitization increases during retirement. In previous analyses, such as Reichling and Smetters (2015), when risk aversion increases it translates into willing to save more in other non-annuitized assets. The logic is straightforward since extra costs can arise at advanced ages and it would be better to buffer them with non-annuitized savings. But in the present model, apart from the individual’s desire to smooth consumption, the individual does not face shocks nor big changes in her beliefs during the last years of her life. As a result, we can see these two motives aligned: a risk averse person would want to use the mortality credit to smooth consumption, and therefore, annuitize more when more risk averse. This effect though is negligible during the working years. These results can be seen in Figure 7. I compare three optimal annuitization paths differentiated only by their CRRA coefficients, baseline $\zeta = 3$, more risk-averse utility with $\zeta = 7$, and less risk-averse utility with $\zeta = 2$. 

![Figure 6: Annuitization with Replacement Ratio Comparison](image)
5 Conclusions

This paper shows how subjective mortality beliefs, as the result of the process of learning actual mortality, influence the demand for annuities in the life-cycle. Mortality beliefs are of central importance given that individuals act on what they believe is the nature of their mortality, and therefore, subjective survival probabilities in this market are essential considering that annuities insure against longevity risk. First I calibrate the parameters of a learning process using subjective survival probabilities data from the HRS. Individuals learn in a Bayesian manner about their actual frailty and in the process form subjective beliefs about it. Subsequently, I use the generated subjective survival probabilities to construct a life-cycle model in which individuals must decide recursively to annuitize or not, highlighting in this manner the subjective mortality role for optimal annuitization.

The present model shows that absent income shocks, or other type of constraints affecting income at advanced ages, optimal annuitization is already incomplete. The baseline results show that the reduction in the demand for annuities at different retirement ages can be on average 22% solely attributable to individuals learning enough about their true mortality chances.

Furthermore, we can see how the noise and initial beliefs in the learning process affect the levels of annuitization. We can conclude from this observation that the individual would annuitize more during retirement years had she more difficulty in learning her actual survival chances in the life-cycle. This then leads us to conclude that knowledge about the true survival chances are important to explain the annuity puzzle because, in most past theoretical models, it has always been assumed that for
optimal annuitization individuals face deterministic survival probabilities, that is, a single perceived
distribution of survival probabilities throughout the life-cycle. In this work I do not only assume
stochastic survival probabilities, as advocated by Reichling and Smetters (2015), but also assume that
the stochastic nature of these survival probabilities reflect the improving perception of the actual
survival chances an individual has.

Additionally, I find that the more risk averse individuals are, the higher their demand for annuities
will be, and finally, I conclude that the replacement ratio has a negligible role on annuitization in this
context.

This work is meant to be complementary to the already large literature about optimal annuitiza-
tion. These findings suggest that a large fraction of annuitization, or the absence of it in the data,
can be explained if we take into account the formation subjective mortality beliefs and the improving
nature of these.

References

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