Credit Market Imperfection, Labor Supply Complementarity, and Global Indeterminacy

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Abstract

The possibility of sunspot equilibria and endogenous cycles due to self-fulfilling expectations are explored in a one-sector overlapping-generations model with credit market imperfection. It is shown that a strategic complementarity between agent’s optimal labor supply decisions may arise when the credit market is imperfect. Multiplicity of steady states and local indeterminacy of equilibrium can arise if the complementarity between agents’ labor supply decisions is sufficiently strong. In some cases, local determinacy can also lead to global indeterminacy in which case stationary sunspot equilibria can be constructed as a randomization between different locally unique equilibrium paths. As a result, the aggregate fluctuations due to self-fulfilling expectations can take place and history-versus-expectations considerations can arise. Escape from underdevelopment trap as well as fall into poverty can become a self-fulfilling prophecy.

Keywords: Borrowing constraint; credit cycles; elastic labor supply; endogenous fluctuations; self-fulfilling expectations.

JEL classification: E32; E44; O16;

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1 Introduction

The idea that imperfections in the credit markets may amplify and propagate shocks to the economy has been suggested by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Information asymmetry between lenders and borrowers creates an agency problem and makes the premium between external and internal finance to be positive. Therefore, an adverse shock to the economy can negatively affect firm’s value of pledgable assets and therefore may create a negative spiral of low investment and low profit. This argument provides a nice explanation for how relatively small and temporary exogenous shocks to the economy may be amplified and become persistent, but falls short for explaining how credit reversal can occur and why periods of expansion may be followed by recessions and vice versa.\footnote{In both models, steady states are stable and any fluctuations could dissipate in the absence of exogenous shocks. Kiyotaki and Moore (1997) demonstrate that the equilibrium dynamics display oscillatory convergence to the steady state, which is the justification behind their “credit cycle” claim.}

Azariadis and Smith (1996), Azariadis and Smith (1998), and others, offer a channel through which expectational indeterminacy can be a source of credit reversal. Changes in beliefs can influence agents’ current choices in such a way that the fluctuations in beliefs become self-fulfilling. Instability of expectations in such environment can be a main source of cyclical volatility and fluctuations of self-fulfilling expectations can cause the transition between two regimes: the so called Walrasian regime of high real activity, substantial lending by the financial institutions, relatively efficient intermediation (due to a low cost of state verification incurred by lenders), and a regime of credit rationing with low real activity, limited credit extension, and relatively inefficient intermediation (due to high costs of state verification). Different regimes in Azariadis and Smith (1996) arise because of the assumptions that (a) there is imperfection in the credit market, and (b) there is an increasing returns to scale in financial intermediation. As a result, low (high) aggregate lending activity implies low (high) intermediation cost incurred from costly state verification. If lenders are “optimistic” about the credit market conditions then there is extensive lending, low intermediation costs, and a large volume of investment. Moreover, in equilibrium, low cost of intermediation justifies the initial optimism. The exact opposite happens in the case of initial “pessimism.” Two different regimes in Azariadis and Smith (1998) arise because of adverse selection problem in credit markets. In case of more aggregate lending, the state verification becomes cheaper for all lenders and vice versa. As a result, adverse selection problems in financing capital goods create credit cycles associated with indeterminacy of equilibrium.

This paper offers a theory how borrowing constraint can induce instability into the dy-
namics even in the absence of an increasing return to scale in financial intermediation and without any cost of state verification. When there is imperfection in the credit market and the labor supply is endogenous then credit reversals can still take place. As a result, recurrent fluctuations can occur even in the absence of any external shock. In the presence of credit market imperfection, aggregate labor supply can play a dual role. High aggregate labor supply is associated with high aggregate saving which on the one hand implies low marginal product of capital and thus low borrowing/lending rate and on the other hand implies more investment in highly productive entrepreneurial activity and thus high return on individual saving. When the former effect is dominant, then the individual and the aggregate saving complement each other. \(^2\) \text{i.e., agents supply more labor only because they expect high aggregate labor supply and high borrowing/lending rate. As a result, the initial expectation about high aggregate labor supply becomes self-fulfilling. Strategic complementarity leads to multiple equilibria, with high-activity equilibria superior to low-activity equilibria. These multiple equilibria simply reflect the fact that if agents anticipate a higher aggregate labor supply then individual agents benefit from supplying more labor. This implies a rise of actual aggregate saving and causes the market to validate the initial optimism.}

When agents incur disutility from supplying labor, then they find balance between current leisure and second period consumption. Low leisure is individually detrimental but contributes to the improvement of net worth of other borrowers because of labor supply complementarity. This in case of binding borrowing constraint generates the aggregate demand spillover, meaning more productive investment and higher return on individual saving. This reinforces the initial low leisure activity by individual workers. The exact opposite is true in case of initial pessimism. This is the mechanism how credit reversal can take place only through self-fulfilling expectations. It is clear that individual labor supply decision will not generate any aggregate demand spillover in the absence of credit market imperfection. This is so because the borrowing constraint would never bind in the absence of the credit market imperfection. As a result, the borrowing/lending rate would always coincide with the marginal product of capital. This implies that the high aggregate labor supply would always create a downward pressure on borrowing/lending rate and thus would discourage individual labor supply. The complementarity between individual and aggregate labor supply would disappear and the credit reversal due to self-fulfilling expectation would not occur. The same is true in the case when agents do not incur any disutility from labor supply. The borrowing/lending rate would still reflect the imperfection on the credit market but this rate would play no role for individual labor.

\(^2\)The standard definition of complementarity is that the optimal strategy of one decision-maker is increasing in the strategies of other similar decision-makers.
supply decisions.

The rest of the paper is organized as follows. In section 2 we outline the model, derive agent’s optimal labor supply decision, and set conditions for equilibrium on the labor and capital markets. In section 3 we specify the dynamical system which describes the evolution of the economy under perfect foresight dynamics. We show the existence of multiple steady states and perform their local stability analysis. In section 4 we consider a parameterized version of the model and analyze the local and global dynamics of the economy. We numerically demonstrate the possibility of (a) heteroclinic connections between different stationary equilibria, and (b) homoclinic bifurcation of a saddle point. In section 5 we summarize the results and conclude.

2 The Model

We consider a discrete time economy populated by an infinite sequence of two period lived overlapping generations. In each period \( t = 0, 1, \ldots \), there are two generations alive, young and old. Each generation is identical in composition, and consists of a continuum of agents of measure one. There is one consumption commodity produced in each period by a large number of identical firms using capital and labor as inputs. The technology of the consumption goods producing firm is described by a constant return to scale production function. Output per worker is \( y_t = f(k_t) \), where \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is the production function in intensive form, \( K_t \) and \( L_t \) are the aggregate supplies of physical capital and labor respectively, and \( k_t = K_t/L_t \) is the capital per worker. Factor markets are competitive and rewards on physical capital and labor are determined by marginal product rule, i.e., \( f'(k_t) \) is the rate of return on one unit of capital and \( w_t = W(k_t) := f(k_t) - k_t f'(k_t) \) is the wage rate. In the following we shall make use even of the function \( \rho(k_t) = k_t f'(k_t) \), the capital income in production. We assume that \( f \) is twice continuously differentiable, that \( f(0) = 0 \), that \( f \) is strictly increasing and strictly concave, and that \( f \) satisfies the usual Inada conditions, \( \lim_{k \downarrow 0} f'(k) = \infty \) and \( \lim_{k \uparrow \infty} f'(k) = 0 \). Produced commodity can be either consumed or invested in capital, which becomes available in the next period. Capital depreciates fully within a period.

The young agent \( i \in [0, 1] \) is endowed with \( \bar{l} > 0 \) units of labor and maximizes the quasi-linear intertemporal utility function

\[
(l_t^i, c_{t+1}^i) \mapsto c_{t+1}^i - \theta u(l_t^i),
\]

where \( l_t^i \in [0, \bar{l}] \) is the first period labor supply, \( c_{t+1}^i \) is next period consumption, \( \theta > 0 \) is the scale parameter, and \( u : [0, \bar{l}] \to \mathbb{R}_+ \) describes the disutility from work. We assume
that \( u \) is twice continuously differentiable, strictly increasing and strictly convex, and that \( u \) satisfies the usual Inada conditions, \( \lim_{l \to 0} u'(l) = 0 \) and \( \lim_{l \to \infty} u'(l) = \infty \).

The young agent has two options to convert the current wage income \( s_t^i = l_t^i w_t \) into the next period consumption. First, she can lend the entire wage income at the competitive credit market at the rate of return \( r_{t+1} \) and consume \( c_{t+1}^i = l_t^i w_t r_{t+1} \) in the second period. Second, she can run a non-divisible investment project\(^3\) which converts one unit of consumption good in period \( t \) into one unit of capital in period \( t + 1 \). If \( s_t^i < 1 \) and the young agent runs an investment project then she needs to borrow \( 1 - s_t^i \). Each project generates \( f'(k_{t+1}) \) units of consumption goods and thus borrower’s second period consumption is \( c_{t+1}^i = f'(k_{t+1}) - (1 - l_t^i w_t) r_{t+1} \). Direct comparison implies that agents would borrow and run an investment project when the following Profitability Constraint is satisfied, \( r_{t+1} \leq f'(k_{t+1}) \).

Suppose that lender’s protection is not perfect and an entrepreneur can hide a \( 1 - \lambda \), \( \lambda \in (0, 1) \), portion of their revenue from financiers. This implies that entrepreneurs can credibly pledge to lenders only a \( \lambda \) fraction of project’s revenue for repayment. As a result, young agents are able to borrow and start the investment project when the following Borrowing Constraint is satisfied, \( (1 - s_t^i) r_{t+1} \leq \lambda f'(k_{t+1}) \). Parameter \( \lambda \) captures the credit market imperfection in a parsimonious way and allows us to investigate the aggregate implication of credit market imperfection. See Matsuyama (2007) for more detailed discussion.\(^4\)

### 2.1 Equilibrium in the Labor Market

In order to find the equilibrium in the labor market, let us first analyze the agent’s optimal labor supply decision. We can work backward. Suppose the young agent becomes a lender in the second period. Then her optimization problem has the following form,

\[
\max_{l_t^i \in [0, \infty]} l_t^i w_t r_{t+1} - \theta u(l_t^i). \tag{2}
\]

\(^3\)Each entrepreneur can run only one investment project which requires a minimum one unit of consumption good for investment.

\(^4\)A large number of studies are devoted to the issues of credit market imperfections and its macroeconomic implications. For example, there is the costly-state-verification approach of Townsend (1979), used by Bernanke and Gertler (1989), Boyd and Smith (1997), Bhattacharya and Chakraborty (2005), and others, the moral hazard approach used by Holmstrom and Tirole (1997) and others, the adverse selection approach used by Hart and Moore (1994), Kiyotaki and Moore (1997) and others. In this paper, we will not argue which of the above stories offer more plausible explanations for credit market imperfections. Instead, we will rely on the reduced form approach of Matsuyama (2000), Matsuyama (2004), and Matsuyama (2007), parameterize the severity of credit market imperfection and analyze its macroeconomic implications.
This with the properties of \( u \) implies the following optimal labor supply decision for lenders\(^5\)

\[
L^s(w_t, r_{t+1}) := (u')^{-1} \left( \frac{w_t r_{t+1}}{\theta} \right). \tag{3}
\]

If the young agent becomes an entrepreneur and borrows in the second period then her current optimization problem has the following form

\[
\max_{l_t \in [0, l]} f'(k_{t+1}) - (1 - l_t w_t) r_{t+1} - \theta u(l_t). \tag{4}
\]

Direct comparison of (2) and (4) shows that when credit allocation is random then all young agents will supply the same amount of labor independent from whether they become borrowers or lenders.

Properties of \( u \) imply that the labor supply curve is strictly increasing with respect to wage rate, and satisfies the boundary conditions: \( L^s(0, r_{t+1}) = 0 \) and \( L^s(\infty, r_{t+1}) = \bar{l} \) for any \( r_{t+1} > 0 \). Labor demand, for a given \( K_t > 0 \), is:

\[
L^d(w_t, K_t) = \frac{K_t}{W^{-1}(w_t)}, \tag{5}
\]

where \( W^{-1} \) denotes the inverse of the wage function. Properties of the production function imply a well-behaved labor demand curve. Labor market clearing wage rate satisfies the following equation

\[
L^d(w_t, K_t) = L^s(w_t, r_{t+1}). \tag{6}
\]

Properties of \( L^s \) and \( L^d \) imply the existence and uniqueness of a labor market clearing wage rate and the market clearing employment.

### 2.2 Equilibrium in the Capital Market

Since young agents supply the same amount of labor, it follows that the individual and aggregate savings are the same, \( s_t^i = s_t \). This implies that the next period capital stock \(^6\) is \( K_{t+1} = s_t \).

Suppose \( s_t \in (0, 1 - \lambda) \). Then the borrowing constraint is satisfied and the rate of return is given by the following equation:

\[
r_{t+1} = \frac{\lambda}{1 - s_t} f'(k_{t+1}). \tag{7}
\]

\(^5\)Monotonicity property of \( (u')^{-1} \) with expression (3) implies that leisure and consumption are gross substitutes and thus labor supply responds positively to increase of wage rate and increase of borrowing/lending rate.

\(^6\)\( s_t \) denotes the aggregate saving and also represents the fraction of agents who borrow and invest. This is so because each investment project requires one unit of final commodity for investment. In the next section we restrict the model parameters so that \( s_t < 1 \) holds for any \( t \).
Since \( r_{t+1} < f'(k_{t+1}) \), it follows from (2) and (4) that all agents would strictly prefer to borrow in the credit market and run an investment project. Equilibrium allocation of credit in this case necessarily involves credit rationing. All agents will apply for credit, but not all credit applications will be fulfilled. Credit is allocated randomly across credit applicants with equal probabilities. The fraction \( s_t \) of the agents will become entrepreneurs, while the rest of the agents are denied credit and thus will become lenders. Each lender will save \( s_t \), and each entrepreneur will borrow \( 1 - s_t \). As a result, total amount of funds borrowed and lent is \( s_t(1 - s_t) \).

If \( s_t \in (1 - \lambda, 1) \) then the profitability constraint is satisfied and the rate of return is given by the following equation:

\[
    r_{t+1} = f'(k_{t+1}).
\]

(8)

When the rate of return coincides with the marginal product of capital then it follows from (2) and (3) that young agents are indifferent between becoming a borrower or a lender.

To summarize, it follows from (7) and (8) that the rate of return depends not only on the marginal product of capital but also on the aggregate savings and is given by

\[
    r_{t+1} = \begin{cases} 
    \frac{\lambda}{1 - s_t} f'(k_{t+1}) & \text{if } s_t < 1 - \lambda \\
    f'(k_{t+1}) & \text{if } s_t \geq 1 - \lambda.
    \end{cases}
\]

(9)

(3) and (9) together imply the complementarity between individual and aggregate labor supply.

### 3 Perfect Foresight Dynamics

Capital and Labor market clearing conditions imply

\[
    K_{t+1} = S(K_t, L_t) \quad \text{and} \quad L_{t+1} = \frac{S(K_t, L_t)}{\xi(K_t, L_t)},
\]

(10)

where \( S(K, L) := LW \left( \frac{K}{L} \right) \) and \( k_1 = \xi(K, L) \) solves the following equation

\[
    \theta Lu' (L) = \begin{cases} 
    \frac{\lambda}{1 - S(K, L)} S(K, L) f'(k_1) & \text{if } S(K, L) < 1 - \lambda \\
    S(K, L) f'(k_1) & \text{if } S(K, L) \geq 1 - \lambda.
    \end{cases}
\]

(11)
Global invertibility of $f'$ with (11) implies that

$$
\xi(K, L) := \begin{cases} 
(f')^{-1} \left[ \frac{1 - S(K, L) \theta L u'(L)}{S(K, L)} \right] & \text{if } S(K, L) < 1 - \lambda \\
(f')^{-1} \left[ \frac{\theta L u'(L)}{S(K, L)} \right] & \text{if } S(K, L) \geq 1 - \lambda.
\end{cases} \quad \text{(12)}
$$

It follows from (10) and (12) that the evolution of the pair $(K_t, L_t)$, under perfect foresight dynamics, is described by the following well defined, two dimensional dynamical system

$$
M : \begin{cases} 
K_{t+1} = m_1(K_t, L_t) \\
L_{t+1} = m_2(K_t, L_t),
\end{cases} \quad \text{(13)}
$$

where

$$
m_1(K, L) := S(K, L) \text{ and } m_2(K, L) := \frac{S(K, L)}{\xi(K, L)}. \quad \text{(14)}
$$

### 3.1 Steady State Analysis

In order to find the steady states of $M$, we solve the following system of equations

$$
\frac{K}{L} = W \left( \frac{K}{L} \right) \quad \text{and} \quad \frac{K}{L} = \xi(K, L). \quad \text{(15)}
$$

First equation of (15) implies that the capital labor ratio $k^*$ at any steady state should satisfy the equation $k = W(k)$.

**Assumption 1** Let $f$ be such that

(a) the function $k \mapsto W(k)$ is strictly decreasing and satisfies boundary conditions

$$
\lim_{k \uparrow \infty} \frac{W(k)}{k} < 1 < \lim_{k \downarrow 0} \frac{W(k)}{k}; \quad \text{(16)}
$$

(b) the function $k \mapsto \rho(k) = k f'(k)$ is non-decreasing;

Assumption 1.(a) implies that $k = W(k)$ admits one interior (non trivial) solution $k^*$. In addition, Assumption 1 restricts the substitution between capital and labor inputs to be bounded from below and thus the aggregate saving, $S$, to be monotonically increasing.
with respect to both arguments. The second equation of (15) implies that the steady state labor should satisfy $\Psi(L) = 0$, where

$$
\Psi(L) := -\frac{\rho(k^*)}{\theta} + \begin{cases} 
\frac{1 - k^*L}{\lambda u'(L)} & \text{if } L < \frac{1 - \lambda}{k^*} \\
u'(L) & \text{if } L \geq \frac{1 - \lambda}{k^*}.
\end{cases}
$$

(17)

Let $\epsilon(l) := \frac{u'(l)}{lu''(l)}$ denotes the elasticity of labor supply (elasticity of $(u')^{-1}$).

**Assumption 2** Let $u$ be such that $l \mapsto \epsilon(l)$ is non-decreasing.

Assumption 2 is technical in nature and restricts the monotonicity of labor supply elasticity. This helps us to minimize the notation and focus solely on the case in which there exists at most three steady states. Many standard functions satisfy Assumption 2. For example, if $u(l) = \frac{\epsilon}{1 + \epsilon} l^{1 + \frac{1}{\epsilon}}$ with $\epsilon > 0$, then $u'(l) = l^{\frac{1}{\epsilon}}$ and the elasticity of labor supply is the constant $\epsilon > 0$.

Let us define the function $H : \left[0, \frac{1}{k^*}\right] \rightarrow \mathbb{R}_+$ as follows

$$
H(L) = \frac{1 - k^*L}{\lambda u'(L)}.
$$

(18)

**Lemma 1** If Assumption 2 is satisfied then $H'(L) = 0$ admits a unique solution $L^c \in \left(0, \frac{1}{k^*}\right)$ which is independent of parameter $\lambda$ and satisfies the equation

$$
\frac{k^*L}{1 - k^*L} \epsilon(L) = 1.
$$

(19)

Lemma 1 implies that if $\lambda < \lambda^c := 1 - k^*L^c$ then $\Psi$ is non-monotonic and thus $\Psi(L) = 0$ can admit at most three solutions. Figure 1 displays all possible configurations of function $\Psi$.

Let us define the functions $\psi_1 : (0, 1] \rightarrow \mathbb{R}_+$ and $\psi_2 : (0, 1] \rightarrow \mathbb{R}_+$ as follows

$$
\psi_1(\lambda) = \frac{\lambda \rho(k^*)}{(1 - k^*L^c)u'(L^c)} \quad \text{and} \quad \psi_2(\lambda) = \frac{\rho(k^*)}{u'(\frac{1 - \lambda}{k^*})}.
$$

(20)

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This Assumption rules out the possibility of endogenous fluctuations due to self-fulfilling expectations to take place in Reichlin (1986). See Section 5 for more discussions.
It is easy to verify that $\psi_1$ and $\psi_2$ satisfy the following equations

$$H(L^c) = \frac{\rho(k^*)}{\psi_1(\lambda)} \quad \text{and} \quad H\left(\frac{1-\lambda}{k^*}\right) = \frac{\rho(k^*)}{\psi_2(\lambda)}.$$  \hspace{1cm} (21)

In other words, when $\theta = \psi_1(\lambda)$ then the function $H$ is tangent to the line $\frac{\rho(k^*)}{\theta}$ at $L = L^c$. When $\theta = \psi_2(\lambda)$ then the equation $H(L) = \frac{\rho(k^*)}{\theta}$ has a solution at $L = \frac{1-\lambda}{k^*}$.

**Proposition 1** Suppose Assumptions 1 and 2 are satisfied and the parameter pair $(\lambda, \theta)$ is fixed. Then:

(a) if $\theta > \psi_2(\lambda)$ then $\Psi(L) = 0$ admits a unique solution $L^*_1 \in (0, L^c)$;

(b) if $\lambda < \lambda^c$ and $\psi_1(\lambda) < \theta < \psi_2(\lambda)$ then $\Psi(L) = 0$ admits three solutions

$$L^*_1 < L^c < L^*_2 < \frac{1-\lambda}{k^*} < L^*_3 = \left(\frac{\rho(k^*)}{\theta}\right)^{-1}.$$

(c) if either $\lambda < \lambda^c$ and $\theta < \psi_1(\lambda)$ or $\lambda > \lambda^c$ and $\theta < \psi_2(\lambda)$ then $\Psi(L) = 0$ admits a unique solution $L^*_3 = \left(\frac{\rho(k^*)}{\theta}\right)^{-1}$.

The intuition behind Proposition 1 is illustrated in Figure 2, where a qualitative representation of the parameter space is given. In region A (which is the one bounded by $\psi_2$ from below) there exists a unique steady state $S^* = (k^*L^*_1, L^*_1)$. In region C (which is the one bounded by $\theta_{\min}$ from below and either by $\psi_1(\lambda)$ or by $\psi_2(\lambda)$ from above) there exists a unique steady state $Q^* = (k^*L^*_3, L^*_3)$. In region B there exists three steady states, $S^*$, $Q^*$, and $E^*$, where $E^* = (k^*L^*_2, L^*_2)$. In region D (which is the one bounded by $\psi_1(\lambda)$ from below and by $\theta_{\min}$ from above) there exist two steady states, $S^*$ and $E^*$, since $Q^*$ is unfeasible. The borrowing constraint is binding at the steady states $S^*$ and $E^*$, while the profitability constraint is binding at the steady state $Q^*$, whenever they exist.

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8It follows from Lemma 2 that Figure 2 is a generic representation of the parameter space.

9A steady state is feasible is the steady state saving is less than unity and thus entrepreneurs need to borrow in the competitive credit market in order to start the investment project.
3.2 Dynamics

We now examine the local stability properties of the steady states. In particular, we analyze how the occurrence of local indeterminacy, local bifurcations, and ultimately the emergence of deterministic fluctuations driven by self-fulfilling expectations depends on the parameter values.

Proposition 2 If Assumptions 1 and 2 are satisfied then

(a) the steady states $S^*$ and $Q^*$, whenever they exist, are always saddles.

(b) the steady state $E^*$, whenever it exists, can be either a source, or a sink.

From the above local stability property we can conclude that the steady states $S^*$ and $Q^*$, whenever they exist, are always locally determinate.\(^{10}\) The steady state $E^*$, whenever it exists, is locally determinate if it is either a saddle or a source. Otherwise, the middle steady state is locally indeterminate. In such a case, for a given capital stock there exists a continuum of perfect foresight trajectories converging to the middle steady state, eventually fluctuating around it.

\(^{10}\)A steady state is locally determinate when in its small neighborhood there exists a unique perfect foresight equilibrium converging to it. In the present model, this means that for a given capital stock there exists a unique value of the control variable - labor supply - leading to the steady state.
To complete the description of the parameter space in Figure 2, we can say that the horizontal line $\theta = \theta_{\text{min}}$ separates the regions where the saddle $Q^*$ from unfeasible becomes feasible (as $\theta$ increases); the branch of the curve $\psi_1(\lambda)$ with $\lambda \leq \lambda_c$ is a saddle-node bifurcation curve, the crossing of which (from below) causes the appearance of the saddle $S^*$ and of the node $E^*$ and the latter may be either unstable or stable, depending on the disutility and production functions and on the associated parameters; the curve $\psi_2(\lambda)$ is a border collision bifurcation curve, at which two steady states merge changing their state from virtual\textsuperscript{11} to non-virtual and vice versa. More precisely, when $\lambda \leq \lambda_c$, at the border collision bifurcation the saddle $Q^*$ merges with $E^*$ and both the fixed become virtual, leaving the saddle $S^*$ as unique steady state; otherwise, when $\lambda > \lambda_c$, at the border collision bifurcation the saddle $Q^*$ merge with the virtual fixed point $S^*$ and immediately after the bifurcation $Q^*$ becomes virtual while $S^*$ is the unique saddle steady state of the map.

Due to the nonlinearity of the model, local stability analysis alone is not sufficient for a full characterization of the model’s behavior. Moreover, drawing conclusions based solely on local analysis can be wrong in general, since the proof of the existence of a perfect foresight path in a neighborhood of a steady state does not rule out the possibility of other bounded trajectories. As shown later, the model under consideration may exhibit global indeterminacy.\textsuperscript{12} This is why conclusions based on global dynamics can be dramatically different from local dynamics.\textsuperscript{13}

The aim of the global analysis is to give a complete description of the invariant sets (attracting and repelling sets, basins of attraction, heteroclinic and homoclinic orbits...) of the model and to investigate the occurrence of bifurcations that cause important qualitative changes in their structure. Such bifurcations may be either local, if they can be analyzed only by means of local information of the map (namely, through the Jacobian matrix) or global, if even global properties of the map are involved in their study.

Among the latter, heteroclinic and homoclinic bifurcations play an important role if associated to the analysis of the (in)determinacy of a steady state, since they cause the appearance of heteroclinic orbits (i.e., orbits connecting distinct steady states) and homoclinic orbits (i.e., orbits connecting a steady state with itself and associated with chaotic

\textsuperscript{11}For piecewise maps, a steady state is virtual when it is a fixed point of a branch of the map but it belongs to a region where such a branch has not to be considered.

\textsuperscript{12}Global indeterminacy means that there exist perfect foresight paths converging to different equilibrium dynamics. That is, different choices of control variables might imply different long run behavior and initial conditions do not necessarily determine to which steady state the economy will eventually converge.

\textsuperscript{13}Our analysis reinforces the concerns expressed by Grandmont and de Vilder (1998), Christiano and Harrison (1999), Pintus and de Vilder (2000), Benhabib and Eusepi (2005),and others
dynamics). Then, in the following numerical example, we shall investigate the occurrence of such bifurcations illustrating the implications they have on the global indeterminacy of the model.

Concerning this, it is worth to observe that the map $M$ defined in (12) is invertible, as can be proved in a straightforward way showing that the system of equations

$$\begin{cases} x = LW \left( \frac{K}{L} \right) \\ y = \frac{x}{\xi(K, L)} \end{cases}$$

has a unique solution with respect to $(K, L)$ for any pair $(x, y) \in \mathbb{R}^2$.

4 Numerical Example

To fix ideas, we consider a parameterized version of the above economy. Suppose that production and marginal disutility functions are:

$$f(k) = k^\alpha \quad \text{and} \quad u'(l) = l^\frac{1}{\epsilon}, \quad (23)$$

where the parameter $\alpha \in (0, 1)$ describes the capital share in production and the parameter $\epsilon > 0$ measures the labor supply elasticity. Assumptions 1 and 2 are automatically satisfied and

$$k^* = (1 - \alpha)^{\frac{1}{1-\alpha}}, \quad f'(k^*) = \frac{\alpha}{1 + \alpha}, \quad \theta_{\text{min}} = \alpha (1 - \alpha)^{\frac{1+\alpha}{\epsilon(1-\alpha)}}, \quad \text{and} \quad \tilde{L} = \infty. \quad (24)$$

It follows from (19) and (20) that

$$L^c = \frac{1}{k^*} \frac{1}{1 + \epsilon}, \quad \lambda^c = \frac{\epsilon}{1 + \epsilon}, \quad \theta^c = \alpha (1 - \alpha)^{\frac{1+\alpha}{\epsilon(1-\alpha)}} (1 + \epsilon)^{\frac{1}{\epsilon}}, \quad (25)$$

$$\psi_1(\lambda) = \frac{\lambda \alpha}{\epsilon} (1 - \alpha)^{\frac{1+\alpha}{\epsilon(1-\alpha)}} (1 + \epsilon)^{\frac{1+\epsilon}{\epsilon}}, \quad \text{and} \quad \psi_2(\lambda) = \alpha (1 - \alpha)^{\frac{1+\alpha}{\epsilon(1-\alpha)}} (1 - \lambda)^{-\frac{1}{\epsilon}}. \quad (26)$$

4.1 Local Indeterminacy

Propositions 1 and 2 associated with expression (25) imply the following corollary

**Corollary 1** If $\alpha < 0.50$, $\epsilon > \frac{\alpha}{1 - 2\alpha}$, $\lambda < \frac{\epsilon}{1 + \epsilon}$, $\theta \in (\psi_1(\lambda), \psi_2(\lambda))$ then the steady state $E^*$ is locally indeterminate.
Proof: Expression (25) with Propositions 1 implies that if \( \lambda < \lambda^c = \frac{\epsilon}{1 + \epsilon} \) and \( \theta \in (\psi_1(\lambda), \psi_2(\lambda)) \) then the middle steady state exists. Expression (37) implies that if \( \alpha < 0.50 \) and \( \epsilon > \frac{\alpha}{1 - 2\alpha} \) then the middle steady state is a sink and thus locally indeterminate.

QED.

Corollary 1 shows that when production and disutility from work are isoelastic then the likelihood for indeterminacy improves as soon as the capital share in production decreases, while the credit market imperfection and the labor supply elasticity increases. By assuming that the capital share in production is \( \alpha = 0.33 \), one obtains from Corollary 1 that a necessary and sufficient condition for local indeterminacy of \( E^* \) (if it exists) is the labor supply elasticity to satisfy \( \epsilon > 0.97 \). In order to get a local indeterminacy with \( \alpha = 0.30 \), one needs the labor supply elasticity to satisfy \( \epsilon > 0.75 \). This implies that the result of local indeterminacy holds for empirically plausible values of labor supply elasticity. As a result, our findings seem to suggest that global indeterminacy can arise in an environment with credit market imperfections because we do not rely upon increasing returns to scale in production or controversial features such as complementarity between consumption and leisure or complementarity between capital/labor inputs.

4.2 Global Indeterminacy

As we have seen in Section 3.2, the coexistence of multiple steady states is a possible issue, then we can claim that the equilibrium paths are (automatically) globally indeterminate, even if the steady states are all locally determinate. Indeed it is possible to set the control variable in order to obtain an equilibrium converging to any steady state, i.e. different bounded trajectories exists. Obviously this is true if we assume that the control variable is freely chosen, without the constraint to be close to a particular steady state (target).

But even when a particular steady state is the target of the controller, and the control variable is set in order to have an initial condition in the neighborhood of the given target, the local determinacy analysis may be not sufficient for a complete understanding of the model behavior, due to the possibility of heteroclinic or homoclinic connections among the different steady states. Indeed, it may occur that in the neighborhood of a

\footnote{Early estimates of labor supply elasticity range between 0.10 – 0.50. For example, based on PSID data, MaCurdy (1981) and Altonij (1986) find the values of the labor supply elasticity to be 0.23 and 0.28, respectively. Killingsworth (1983) finds US labor supply elasticity to equal to 0.40. More recent estimates of labor supply elasticity range between 0.60 – 3.00. For example, Angrist (1991), Gourinchas and Parker (2002), Imai and Keane (2004), and Laitner and Dan (2005), present estimates that range from 0.60 to 0.80, from 0.70 to almost 2.00, from 3.00 to almost 4.00, and around 0.90 respectively.}
local determinate steady state there exist bounded trajectories converging to a different steady state, if a heteroclinic connection between them exists. On the other hand, a homoclinic connection, as it is well known, implies the existence of chaotic dynamics. Then global indeterminacy around the steady state arises, due to the infinitely many bounded trajectories existing in the neighborhood of the homoclinic saddle.

Therefore, in this section we stress the difference among local and global (in)determinacy when the model is restricted to a neighborhood of a target steady state. In particular we illustrate by numerical simulations how local determinacy of target equilibrium may be associated with global indeterminacy around it, pointing out the cases where heteroclinic and homoclinic connections occur. A similar study has been performed even in Agliari and Vachadze (2010), where a particular marginal work disutility has been considered. The present study strengthen the results obtained in that paper, since here we consider a more standard work disutility and obtain even more complex situations.

In the following we set $\alpha = 0.33$ and $\lambda = 0.2$ and consider two different values of $\varepsilon$, specifically $\varepsilon = 0.5$ and $\varepsilon = 1$, in order to cover both situations of all locally determinate steady states and the existence of a locally indeterminate steady state. As bifurcation parameter we consider $\theta$.

### 4.2.1 Heteroclinic connections

Let us start with the case $\varepsilon = 0.5$, considered as an illustrative example of the case in which only locally determined steady states exist.

From Proposition 1 we obtain that when $\theta \in (\theta_{sn}, \theta_{bcb})$ the middle steady state $E^*$ exists (see Figure 3). Moreover, since the assumption on $\varepsilon$ of Corollary 1 is not satisfied, such a steady state is locally determinate. When $\theta \in (\theta_{min}, \theta_{sn})$ then the steady state $Q^*$ exists and it is a saddle. At $\theta = \theta_{sn}$ a saddle-node bifurcation occurs causing the appearance of a repelling node, $E^*$, and a saddle, $S^*$. As $\theta$ increases $E^*$ turns into a repelling node and persists until $\theta < \theta_{bcb}$. At the border collision bifurcation value $\theta = \theta_{bcb}$ the steady states $E^*$ and $Q^*$ merge, both belonging to the constraint $S(K, L) = 1 - \lambda$. Immediately after such a bifurcation, only the steady state $S^*$ survives, the steady states $E^*$ and $Q^*$ becoming virtual fixed points. Then, from the local stability analysis we can conclude that the steady states are all locally determinate for any value of the parameter $\theta$.

The two local bifurcations described above have important consequences even on the global determinacy of the different steady states, as we can argue from Figure 4 where two phase portraits are depicted. In such a figure, we observe that, apart the three steady states and the stable manifolds of the saddle points, $W_s(S^*)$ and $W_s(Q^*)$, all
the equilibrium paths (depicted in grey) are unbounded. Figure 4.(a) has been obtained immediately after the saddle-node bifurcation and three steady states coexist. Moreover, as the behavior of the stable manifold of $S^*$ reveals, a heteroclinic connection between $S^*$ and $E^*$ exists, made up by a branch of $W_s(S^*)$ that issues from the repelling focus $E^*$. It comes as no surprise that such a connection exists, since the figure has been obtained at a value very close to $\theta_{sn}$. But its existence entails global indeterminacy around the locally determinate steady state $E^*$. This means that in a neighborhood of this steady state (excluding $E^*$ itself), different bounded equilibrium paths exist and the control variable $L$ can be chosen so that the economy converges to $S^*$. Then, we can say that the saddle-node bifurcation, although it gives rise to two locally determinate steady states, causes the global indeterminacy of the model restricted to a neighborhood of the repelling fixed $E^*$.

The effect of the border collision bifurcation is illustrated in Figure 4.(b). As we said, when $\theta$ is larger than $\theta_{bcb}$ only the saddle $S^*$ survives as steady state and it may\textsuperscript{15} be globally determinate. The existence of a unique globally determined stationary equilibrium persists even if $\theta$ is further increased and it occurs when $\theta < \theta_{sn}$, as well, involving in such a case the saddle $Q^*$.

\textsuperscript{15}Due to the nonlinearity of the model, some different repelling set may coexist with $S^*$, not detectable through numeric simulations.
Figure 4: Global indeterminacy (panel a) and global determinacy (panel b). In the region $U$ the trajectories are unfeasible, since $S(K_t, L_t) > 1$ at some $t$.

The global indeterminacy around $E^*$ may become even more complex, since considering values of $\theta$ belonging to the range $(\theta_{sn}, \theta_{bcb})$, even heteroclinic connections involving the saddle $Q^*$ may be picked out, as shown in Figure 5. Indeed in Figure 5.(a), not only $W_s(S^*)$ but also $W_s(Q^*)$ comes out of the repelling focus $E^*$. Such a situation is the final outcome of a heteroclinic bifurcation involving a branch of $W_s(Q^*)$ and one branch of the unstable manifold of $S^*$, $W_u(S^*)$, very close each other in Figure 5.(b). We detect the occurrence of such a global bifurcation by looking at the two involved invariant sets which have changed their mutual position. As a consequence of the heteroclinic bifurcation, we obtain that in the neighborhood of the locally determinate stationary equilibrium $E^*$ we have bounded equilibrium paths that can reach anyone of the steady state. Then even when the economy is initially located close to the middle steady state, the agents may coordinate their expectations on a equilibrium path leading either to the low steady state or to the high one. Moreover, being $E^*$ a repelling focus, the economy may fluctuate around the middle steady state before reaching one of the saddle fixed points (see the zoom in Figure 5.(b), white square). In such a case it may not be possible to understand immediately the effect of the expectation coordination. Furthermore, even when the heteroclinic bifurcation develops (that is, when the two involved branches $W_s(Q^*)$ and $W_u(S^*)$ intersect) of even the saddle $S^*$ becomes globally indeterminate since in a neighborhood of it there are equilibrium path converging to the saddle $Q^*$, as explained in Agliari and Vachadze (2010).
Figure 5: Indeterminacy of the stationary equilibrium $E^*$. 

4.2.2 Homoclinic bifurcation

In order to consider a situation in which the middle steady state is locally indeterminate we set $\varepsilon = 1.00$.

Figure 6: Parameter Space when $\varepsilon = 1.00$
From Proposition 1, we have that if \( \theta \in (\theta_{sn}, \theta_{bc}) \) the middle steady state exists (see Figure 6) and it is locally indeterminate since Corollary 1 holds. Moreover, when \( \theta < \theta_{\text{min}} \) the steady state \( Q^* \) is such that the saving function evaluated at it is larger than 1, and consequently it is not feasible. The two bifurcation values are associated with changes analogous to the ones described in the previous subsection, with the unique difference that the saddle-node bifurcation gives rise to a stable node, \( E^* \), and a saddle, \( S^* \). When \( \theta \) is slightly larger than \( \theta_{sn} \) a heteroclinic connection exists between the two just appeared steady state, made up by the unstable manifold \( W_u(S^*) \) that reaches the stable fixed point, as shown in Figure 7.(a). In such a Figure \( E^* \) is turned into an attracting focus and the stable manifold \( W_s(S^*) \) is the boundary of its basin of attraction. Hence, now we have global indeterminacy around the stationary equilibrium \( S^* \), since in its neighborhood there also exist infinitely many bounded trajectories converging to the middle steady state.

As the parameter \( \theta \) is further increased, a homoclinic bifurcation of the saddle \( S^* \) occurs, indeed a transversal crossing of one branch of the unstable manifold of \( S^* \) with the boundary of the basin of attraction \( E^* \) (made up by the stable manifold of \( S^* \)) can be observed (see Figure 7.(b)). As it is well known, a homoclinic bifurcation is originated by a first tangential contact between two branches of the stable and unstable manifolds and closed by a second tangential contact occurring at opposite side with respect to the previous one. In the parameter range between the two tangential contacts, a transversal crossing of the two involved branches takes place, as in Figure 7.(b). Such a sequence is called a homoclinic tangle and it implies the existence of a chaotic repellor in the portion of the phase plane where the manifolds intersect each other. That is, an invariant set exists, made up by infinitely many (countable) repelling cycles and uncountable aperiodic trajectories and the restriction of the map to this invariant set is chaotic, that is, it admits a closed invariant set with dense periodic points and it is topologically transitive.

Hence, in the parameter range where the homoclinic tangle exists, around the locally determinate saddle \( S^* \) it is possible to find infinitely many bounded trajectories belonging to either some repelling cycle or to their stable sets and the equilibrium path may also fluctuate chaotically. And the economy can escape from the low steady state if expectations are coordinate.

Another important feature of the nonlinear models is that stationary cyclical bounded equilibrium path may exist. An example is shown in Figure 8.(a), where a stable cycle \( C \) of period 8 coexists with the stable focus \( E^* \). Such a locally indeterminate cycle has appeared through a saddle-node bifurcation together with a saddle cycle \( N \) of the same period. The latter period-8 cycle is locally determinate, but immediately after the bifurcation a heteroclinic connection exists. Indeed one branch of the unstable manifold of \( N \) reaches the period 8 cycle, while the stable manifold \( W_s(N) \) gives the basin boundary
of $C$. Then we can conclude that in the neighborhood of $N$ there exists infinitely many bounded equilibrium paths converging to $C$, and global indeterminacy around the locally determinate period 8 cycle is established. Moreover, the saddle $S^*$ can be chaotic, as illustrated in Figure 8.(b) where the stable and unstable sets are shown, it may also happen that the invariant sets of the saddle period 8 cycle intersect those of $S^*$, so that even a heteroclinic connection between the two locally determinate equilibria, $S^*$ and $N$, may exist.

Figure 7: Heteroclinic and homoclinic connections. Only one of the two differentiable maps defining $M$ is involved.
5 Summary and Conclusions

The above considered economy will have a unique and determinate steady state in the absence of credit market imperfection. This is in contrast to the finding of Reichlin (1986), who obtains that the endogenous fluctuations due to self-fulfilling expectations can emerge in an OLG economy with constant returns to scale production technology and elastic labor supply. The mechanism through which the endogenous fluctuations is possible in Reichlin’s model is different from one considered in this paper and follows from the assumption of low substitutability between capital and labor inputs. Standard preferences imply that labor supply responds positively to the wage and the borrowing/lending rate (the same is true in this paper as well). Under the assumption of neoclassical technology, wage and the borrowing/lending rate are negatively correlated. Consider a dynamic path along which the wage rate and the capital stock are expected to rise. Along this path, borrowing/lending is expected to fall and thus the relative price of future consumption in terms of leisure to rise. As a result, labor supply decreases, aggregate saving increases, and the expectation about the high next period capital stock will become self-fulfilling. The exact opposite happens along a dynamic path along which agents expect the capital stock to fall.

Reichlin (1986) demonstrates that endogenous cycles are possible when the elasticity of substitution between capital and labor is lower than the capital share in production, which in turn should be smaller than one half. This implies that one cannot obtain endogenous cycles with an isoelastic production function.
stock to fall. This mechanism relies on the assumption that capital and labor have a low degree of substitutability; otherwise a falling labor supply would imply lower aggregate saving (as considered in this paper) and lower next period capital stock. Reichlin (1986) demonstrates that technology may play a role in generating deterministic cycles. However, Grandmont and de Vilder (1998) show that for reasonable values of other parameters, the elasticity of substitution between capital and labor cannot exceed 0.03 in order to obtain endogenous cycles in Reichlin’s model. In this paper, we do not rely on assumption of sufficiently low substitutability of capital and labor inputs. Instead, we consider a numerical example and demonstrate that the proposed channel of endogenous cycles works even with isoelastic production technology (for which the elasticity of input substitution is unity).

In contrast to Reichlin (1986), this paper contributes to the sunspots literature by providing a model, based upon strategic complementarities that can produce sunspot equilibria without relying on restrictive assumption about production and utility functions. We offer a two sided feedback mechanism through which credit reversal can occur. Imperfection in the credit market implies binding borrowing constraint and causes self-fulfilling expectations to become a main factor for determining the future evolution of the economy. Results obtained in this paper can explain why countries with either high or low credit market imperfections can converge monotonically to low and high steady states respectively, while countries with intermediate level of credit market imperfection can fluctuate endogenously.

The model considered in this paper can be generalized by allowing the first period consumption and by relaxing the assumption about the quasi-linear utility function. Under this more general setup one can investigate whether the occurrence of local and global indeterminacy becomes more or less likely and whether indeterminacy still holds for empirically plausible parameter values.\textsuperscript{17}

\textsuperscript{17}Cazzavillan and Pintus (2004) show that indeterminacy is more likely to occur in Reichlin’s (1986) model when agents have quasi-linear utility function and they do not consume while young.
6 Appendix

Proof of Lemma 1: After taking a natural log of both sides of (18) and then differentiating it, we obtain that

\[
\frac{LH'(L)}{H(L)} = \frac{1}{\epsilon(L)} - \frac{k^* L}{1 - k^* L}.
\]  

(27)

Thus solving \(H'(L) = 0\) is equivalent to solving (19), the left hand side of which is strictly increasing due to Assumption 2. The left hand side of (19) also satisfies the boundary conditions

\[
\lim_{L \downarrow 0} \frac{k^* L}{1 - k^* L} \epsilon(L) = 0 \quad \text{and} \quad \lim_{L \uparrow 1} \frac{k^* L}{1 - k^* L} \epsilon(L) = \infty.
\]  

(28)

These with the intermediate function theorem guarantees the existence and uniqueness of one critical point (which is the point of local maximum) of \(H\). Assumption 2 guarantees the strict monotonicity of the left hand side of (27) and thus existence and uniqueness of \(L^c\) solving \(H'(L) = 0\). If \(\lambda < \lambda^c\) \(\iff\) \(\epsilon \left( \frac{1 - \lambda}{k^*} \right) \frac{1 - \lambda}{\lambda} > 1 \iff H' \left( \frac{1 - \lambda}{k^*} \right) < 0 \iff L^c \in \left( 0, \frac{1 - \lambda}{k^*} \right)\).

QED.

Lemma 2 (a) \(\psi_1\) is a linear function with \(\psi_1(0) = 0\) and \(\psi_1(1) > \theta_{\text{min}}\);

(b) \(\psi_2\) is a strictly increasing function with \(\psi_2(0) = \theta_{\text{min}}\) and \(\psi_2(1) = \infty\);

(c) \(\psi_1\) and \(\psi_2\) tangent each other at \(\lambda^c := 1 - k^* L^c\) and \(\psi_1(\lambda) < \psi_2(\lambda)\) for any \(\lambda \neq \lambda^c\).

Proof: (a) and (b) are trivial.

(c) \(L^c\) is a point of global maximum of \(H\) and \(H\) is strictly decreasing with respect to \(\lambda\). If \(\lambda \neq \lambda^c = 1 - k^* L^c\) \(\iff\)

\[
\psi_2(\lambda) H \left( \frac{1 - \lambda}{k^*} \right) = \psi_1(\lambda) H \left( L^c \right) > \psi_1(\lambda) H \left( \frac{1 - \lambda}{k^*} \right)
\]  

(29)

\(\iff\) \(\psi_2(\lambda) < \psi_1(\lambda)\). In addition,

\[
\frac{\lambda \psi'_2(\lambda)}{\psi_2(\lambda)} = 1 \quad \text{and} \quad \frac{\lambda \psi'_1(\lambda)}{\psi_1(\lambda)} = \frac{\lambda}{1 - \lambda} \epsilon \left( \frac{1 - \lambda}{k^*} \right).
\]  

(30)

(29) and (30) imply the claim of the lemma.

QED.
**Proof of Proposition 1:**

(a) If \( \theta > \psi_2(\lambda) \) then \( \Psi(0) < 0, \Psi(L^c) > 0 \), and \( \Psi \left( \frac{1 - \lambda}{k^*} \right) > 0 \) implying the existence and uniqueness of a solution of \( \Psi(L) = 0 \) on the interval \((0, L^c)\);

(b) If \( \lambda < \lambda^c \) then \( \Psi \) is non monotonic. If in addition, \( \psi_1(\lambda) < \theta < \psi_2(\lambda) \) then \( \Psi(0) < 0, \Psi \left( \frac{1 - \lambda}{k^*} \right) < 0 \) and \( \Psi \left( \frac{1}{k^*} \right) > 0 \) implying the existence three solutions of \( \Psi(L) = 0 \) on the intervals, \((0, L^c), \left( L^c, \frac{1 - \lambda}{k^*} \right), \) and \( \left( \frac{1 - \lambda}{k^*}, \frac{1}{k^*} \right) \) respectively.

(c) If \( \lambda < \lambda^c \) then \( \Psi \) is non monotonic. If in addition, \( \theta < \psi_1(\lambda) \) then \( \Psi(0) < 0, \Psi \left( \frac{1 - \lambda}{k^*} \right) < 0, \Psi \left( \frac{1}{k^*} \right) < 0 \), and \( \Psi \left( \frac{1}{k^*} \right) > 0 \) implying the existence of a unique solution of \( \Psi(L) = 0 \) on the intervals, \( \left( \frac{1 - \lambda}{k^*}, \frac{1}{k^*} \right) \).

If \( \lambda > \lambda^c \) then \( \Psi \) is monotonic. If in addition, \( \theta < \psi_2(\lambda) \) then \( \Psi(0) < 0, \Psi \left( \frac{1 - \lambda}{k^*} \right) < 0, \) and \( \Psi \left( \frac{1}{k^*} \right) > 0 \) implying the existence a unique solution of \( \Psi(L) = 0 \) on the intervals, \( \left( \frac{1 - \lambda}{k^*}, \frac{1}{k^*} \right) \).

QED.

**Proof of Proposition 2:**

In order to prove the proposition we adapt the following strategy: (a) evaluate the trace \( T^* \) and the determinant \( D^* \) at any steady state; (b) show that the determinant of the Jacobian matrix is always positive; and (c) demonstrate that the sign of \( 1 - T^* + D^* \) depends on the slope of \( H \) evaluated at a given steady state.

Jacobian matrix at any given steady state, \((K^*, L^*)\), is

\[
J(K^*, L^*) = \begin{pmatrix} m_{11}(K^*, L^*) & m_{12}(K^*, L^*) \\ m_{21}(K^*, L^*) & m_{22}(K^*, L^*) \end{pmatrix}.
\] (31)

(10) and (12) imply that

\[
m_{11}(K^*, L^*) = W''(k^*) \quad \text{and} \quad m_{12}(K^*, L^*) = W(k^*) - k^*W'(k^*),
\] (32)

and

\[
\begin{align*}
m_{21}(K^*, L^*) &= \frac{1}{k^*} \left( W'(k^*) - \frac{K^*\xi_1(K^*, L^*)}{\xi(K^*, L^*)} \right) \\ m_{22}(K^*, L^*) &= 1 - W'(k^*) - \frac{L^*\xi_2(K^*, L^*)}{\xi(K^*, L^*)}
\end{align*}
\] (33)
(32) and (33) imply that the Trace and Determinant of the Jacobian matrix evaluated at any steady state are

\[
\begin{align*}
T^* &= 1 - \frac{L^* \xi_2(K^*, L^*)}{\xi(K^*, L^*)} \\
D^* &= \frac{K^* \xi_1(K^*, L^*)}{\xi(K^*, L^*)} - W'(k^*) \left( \frac{K^* \xi_1(K^*, L^*)}{\xi(K^*, L^*)} + \frac{L^* \xi_2(K^*, L^*)}{\xi(K^*, L^*)} \right).
\end{align*}
\]  
\tag{34}

(12) implies that

\[
-\frac{k^* f''(k^*) K^* \xi_1(K^*, L^*)}{f'(k^*) \xi(K^*, L^*)} = \begin{cases} 
\frac{W'(k^*)}{1 - k^* L^*} & \text{if } L^* < \frac{1 - \lambda}{k^*} \\
W'(k^*) & \text{if } L^* \geq \frac{1 - \lambda}{k^*},
\end{cases}
\]  
\tag{35}

and

\[
-\frac{k^* f''(k^*) L^* \xi_2(K^*, L^*)}{f'(k^*) \xi(K^*, L^*)} = \begin{cases} 
\frac{k^* L^*}{1 - k^* L^*} - \frac{W'(k^*)}{1 - k^* L^*} - \frac{L^* u''(L^*)}{u'(L^*)} & \text{if } L^* < \frac{1 - \lambda}{k^*} \\
-W'(k^*) - \frac{L^* u''(L^*)}{u'(L^*)} & \text{if } L^* \geq \frac{1 - \lambda}{k^*}.
\end{cases}
\]  
\tag{36}

(34), (35), and (36) imply that

\[
D^* = f'(k^*) \left( 1 + \frac{1}{\epsilon(L^*)} \right) > 0,
\]  
\tag{37}

and

\[
1 - T^* + D^* = f'(k^*) \frac{1 - W'(k^*)}{W''(k^*)} \times \begin{cases} 
\frac{k^* L^*}{1 - k^* L^*} - \frac{1}{\epsilon(L^*)} & \text{if } L^* < \frac{1 - \lambda}{k^*} \\
-\frac{1}{\epsilon(L^*)} & \text{if } L^* \geq \frac{1 - \lambda}{k^*}.
\end{cases}
\]  
\tag{38}

Since \(W'(k^*) \in (0, 1)\), it follows from (38) that both steady states, \(L_1^*\) and \(L_3^*\), whenever they exist, are always saddles and thus locally determinate. In contrast, the steady state \(L_2^*\), whenever it exists, can be either a source, or a sink.

\[\text{QED.}\]
References


