

CUNY GC Mathematics Program
Spring 2010 Algebra Qualifying Exam
Time: 3 hours

Instructions: You will receive full or partial credit for your choice of *six* out of the following ten problems. On this sheet, indicate by checking the appropriate boxes for which six problems you wish to receive credit. Justify all answers and state clearly any major theorems you use. When you are finished, assemble your solutions with this sheet on top.

□ 1. Let G be a finite group. For each element $x \in G$, denote $Z_x = \{y \in G \mid xy = yx\}$ the centralizer of x . The class formula for G states that

$$|G| = \sum_{x \in S} (G : Z_x),$$


where S is a subset of G containing exactly one element from each of the conjugacy classes of G .

- (a) Prove the class formula.
- (b) Assume that G is a nontrivial p -group. Show that G has nontrivial center.

□ 2. Let p be a prime number, let \mathbb{F}_p be the finite field with p elements, and let k/\mathbb{F}_p be a finite extension.

- (a) Show that $|k| = p^n$ for some integer $n \geq 1$.
- (b) Show that k/\mathbb{F}_p is a Galois extension.
- (c) Show that $\text{Gal}(k/\mathbb{F}_p)$ is cyclic.

□ 3. Let A, B, C be groups, written multiplicatively, and let $1 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 1$ be a short exact sequence. This sequence *splits* if there exists a group homomorphism $\phi : C \rightarrow B$ such that $g \circ \phi = \text{Id}_C$.

- (a) Prove that if the sequence splits then $B \cong A \times C$. 
- (b) Let Γ_3 be the cyclic group of order 3 and let Γ_2 be the cyclic group of order 2. Find all groups G such that there exists a short exact sequence of the form $1 \rightarrow \Gamma_3 \xrightarrow{f} G \xrightarrow{g} \Gamma_2 \rightarrow 1$. For which of these groups G does there exist a split short exact sequence as shown?

□ 4. Let ζ be a primitive 9-th root of unity in \mathbb{C} .

- (a) Find the minimal polynomial of ζ over \mathbb{Q} and show that it is irreducible.
- (b) Calculate the group $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and carefully prove your answer. Is $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ cyclic?
- (c) Let $K = \mathbb{Q}(\zeta) \cap \mathbb{R}$. Find an element $\alpha \in \mathbb{R}$ such that $K = \mathbb{Q}(\alpha)$, and calculate $[K : \mathbb{Q}]$. Is K/\mathbb{Q} a Galois extension?

□ 5. Let p be a prime number, let \mathbb{F}_p be the finite field with p elements, let $n \geq 2$ be an integer, and let $\text{GL}_n(\mathbb{F}_p)$ be the group, under matrix multiplication, of nonsingular $n \times n$ matrices over \mathbb{F}_p . Let H be the set of $n \times n$ matrices with only zeros below the main diagonal, only ones along the main diagonal, and arbitrary elements of \mathbb{F}_p above the main diagonal. Prove that H is a p -Sylow subgroup of $\text{GL}_n(\mathbb{F}_p)$.

□ 6. Let G be a finite group and let k be a field. A *one-dimensional character* of G over k is a group homomorphism $\chi : G \rightarrow k^\times$.

- (a) Show that G has only finitely many one-dimensional characters over k .
- (b) Let χ_1, \dots, χ_n be the distinct one-dimensional characters of G over k . Prove that they are linearly independent over k , in the sense that if $a_1, \dots, a_n \in k$ and if $a_1\chi_1 + \dots + a_n\chi_n$ is the identically zero function on G , then $a_1 = \dots = a_n = 0$.

□ 7. Let X be a nonempty finite set, let k be a field, and let $R(X, k)$ be the set of functions $f : X \rightarrow k$. View $R(X, k)$ as a ring under the pointwise addition and multiplication operations.

- (a) Find all ideals of $R(X, k)$.
- (b) Find all maximal ideals of $R(X, k)$.

□ 8. A *Euclidean function* on an integral domain R is a nonnegative-integer-valued function N on $R \setminus \{0\}$ with the following property: for all $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = qb + r$ and either $r = 0$ or $N(r) < N(b)$.

(a) Prove that if R is an integral domain and if N is a Euclidean function on R , then R is a principal ideal domain.

(b) The *ring of Gaussian integers* is $\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}\}$, where $i^2 = -1$. Prove that $N(x + yi) = x^2 + y^2$ is a Euclidean function on $\mathbb{Z}[i]$, and so $\mathbb{Z}[i]$ is a principal ideal domain.

□ 9. Let R be an integral domain which contains a field k as a subring. Show that if R is finitely generated as a k -vector space, then R is a field.

□ 10. Let k be an algebraically closed field and let $R = k[X_1, \dots, X_n]$. Let k^n be the set of all n -tuples (a_1, \dots, a_n) of elements of k . For every set S of polynomials in R , the *affine variety* associated with S is the set

$$V(S) = \{(a_1, \dots, a_n) \in k^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in S\}.$$

(a) Prove that if S_1 and S_2 are subsets of R and if $S_1 \subseteq S_2$, then $V(S_2) \subseteq V(S_1)$.

(b) Let S be a nonempty subset of R , and let I be the ideal generated by S . Prove that $V(S) = V(I)$.

(c) Prove that if I and J are ideals in R , then $V(I) \cup V(J) = V(I \cap J)$.

(d) Prove that if P is a prime ideal of R , then

$$P = \{f \in R \mid f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in V(P)\}.$$