1. Rings of Fractions

1.0. Rings and algebras

(1.0.1) All the rings considered in this work possess a unit element; all the modules over such a ring are assumed unital; homomorphisms of rings are assumed to take unit element to unit element; and unless explicitly mentioned otherwise, all subrings of a ring A are assumed to contain the unit element of A. We generally consider commutative rings, and when we say ring without specifying, we understand this to mean a commutative ring. If A is a not necessarily commutative ring, we consider every A module to be a left A-module, unless we expressly say otherwise.

(1.0.2) Let A and B be not necessarily commutative rings, $\phi : A \to B$ a homomorphism. Every left (respectively right) B-module M inherits the structure of a left (resp. right) A-module by the formula $a.m = \phi(a).m$ (resp $m.a = m.\phi(a)$). When it is necessary to distinguish the A-module structure from the B-module structure on M, we use $M_{[\phi]}$ to denote the left (resp. right) A-module so defined. If L is an A module, a homomorphism $u : L \to M_{[\phi]}$ is therefore a homomorphism of commutative groups such that $u(a.x) = \phi(a).u(x)$ for $a \in A$, $x \in L$; one also calls this a ϕ-homomorphism $L \to M$, and the pair (u,ϕ) (or, by abuse of language, u) is a di-homomorphism from (A, L) to (B, M). The pair (A,L) made up of a ring A and an A module L therefore form the objects of a category where the morphisms are di-homomorphisms.

(1.0.3) Under the hypotheses of (1.0.2) if $\Im$ is a left (resp. right) ideal in A, we use $B\Im$ (resp. $\Im B$) to denote the left (resp right) ideal $B\phi(\Im)$ (resp $\phi(\Im)B$) of B generated by $\phi(\Im)$; this is also the image of the canonical homomorphism $B \otimes_A I \to B$ (resp. $\Im \otimes_A B \to B$) of left (resp right) B modules.

(1.0.4) If A is a (commutative) ring, and B is a not necessarily commutative ring, the data defining the structure of an A-algebra on B is equivalent to the data describing a homomorphism of rings $\phi : A \to B$ such that $\phi(A)$ is contained in the center of B. For each ideal $\Im$ of A, $\Im B = B\Im$ is a doublesided ideal of B, and for each B module M, $\Im M$ is also a B-module via $(B\Im)M$.

(1.0.5) We do not review the notions of module of finite type and (commutative) algebra of finite type; to say an A module M is of finite type signifies the existence of an exact sequence $A^p \to M \to 0$. One says an A module M admits a finite presentation if it is isomorphic to the cokernel of a homomorphism $A^p \to A^q$; equivalently, there exists an exact sequence $A^p \to A^q \to M \to 0$. Note that over a noetherian ring, every A module admits a finite presentation.

We recall that an A algebra B is said to be integral over A if every element of B is the root in B of a monic polynomial with coefficients in A; recall that this is the same as saying that every element of B is contained in a subalgebra of B that is of finite type as an A module. If this is the case, and B is commutative, the subalgebra of B generated by a finite subset is an A module of finite type; in order that the commutative algebra B be integral and of finite type over A, it suffices to show that B is is of finite type as an A module; one then says that B is an integral finite A-algebra (or simply a finite A-algebra if no confusion will result.) One observes that in making these definitions, one does not require that the homomorphism $A \to B$ defining the structure of the A algebra be injective.

(1.0.6) A ring is an integral domain if the product of a finite family of elements $\neq 0$ is $\neq 0$; it is equivalent to say that $0 \neq 1$ and the product of two elements $\neq 0$ is not zero. A prime ideal in a ring A is an ideal $\wp$ such that $A/\wp$ is an integral domain; this requires $\wp \neq A$. For a ring A to contain a prime ideal, it is necessary and sufficient that $A \neq \emptyset$.

(1.0.7) A ring A is local if there exists only one maximal ideal; which is then the complement of invertible elements and contains every ideal $\neq A$. If A and B are local rings, $\mathfrak{m}$ and $\mathfrak{n}$ are their respective maximal ideals, then one says a homomorphism $\phi : A \to B$ is local if $\phi(\mathfrak{m}) \subset \mathfrak{n}$ (or equivalently, $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$). On passing to quotients, such a homomorphism defines a monomorphism from the residual field $A/\mathfrak{m}$ to the residual field $B/\mathfrak{n}$. The composition of two local homomorphisms is a local homomorphism.

1.1. Radical of an ideal. Nilradical and radical of a ring

(1.1.1) Let $\mathfrak{a}$ be an ideal in a ring A; the radical of A, denoted $\Re(\mathfrak{a})$, is the set of all $x \in A$ such that $x^n \in \mathfrak{a}$ for some integer $n > 0$; this is an ideal containing $\mathfrak{a}$. One has $\Re(\Re(\mathfrak{a})) = \Re(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ implies $\Re(\mathfrak{a}) \subset \Re(\mathfrak{b})$; and the radical of a finite intersection of ideals is the intersection of their radicals. If ϕ is a homomorphism from a ring A' to A, one has $\Re(\phi^{-1}(\mathfrak{a})) = \phi^{-1}(\Re(\mathfrak{a}))$ for every ideal $\mathfrak{a} \subset A$. For an ideal to be the radical of an ideal, it is necessary and sufficient that it be the intersection of prime ideals. The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which are minimal among those which contain $\mathfrak{a}$; if A is noetherian, there are a finite number of such ideals. (1.1.2) Recall that the radical $\Re(A)$ of a (not necessarily commutative) ring A is the intersection of the maximal left ideals of A (and also the intersection of the maximal right ideals) The radical of $A/\Re(A)$ is 0.

1.2. Modules and rings of fractions

(1.2.1) One says a subset S of a ring A is multiplicative if $1 \in S$ and if the product of two elements of S are in S. The most important examples are: 1) the set S_f of powers f^n ($n \geq 0$) of an element $f \in A$; and 2) the complement $A - \wp$ of a prime ideal $\wp$ in A.

(1.2.2) Let S be a multiplicative subset of a ring A, and M an A-module; in the set $M \times S$, the relation between pairs (m_1, s_1), (m_2, s_2) :

$$<< \text{there exists } s \in S \text{ such that } s(s_1 m_1 - s_2 m_2) = 0 >>$$

is an equivalence relation. One denotes by $S^{-1}M$ the quotient of $M \times X$ by this relation, and m/s denotes the canonical image in $S^{-1}M$ of the pair (m,s); one calls the mapping M to $S^{-1}M$ given by $i_M^S : m \to m/1$ (also denoted i^S) the canonical mapping. This mapping is not in general surjective, and the kernel is the set of $m \in M$ such that there exist $s \in S$ such that $sm = 0$. On $S^{-1}M$ one defines the structure of an additive group by

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/s_1 s_2)$$

. (One verifies that this is independent of the chosen expression of elements of $S^{-1}M$). On $S^{-1}A$ one also defines a multiplication by $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally one defines an action of $S^{-1}A$ on $S^{-1}M$ by $(a/s)(m/s') = (am)/(ss')$. One verifies that $S^{-1}A$ has the structure of a ring (called the ring of fractions of A with denominators in S) and $S^{-1}M$ has the structure of an $S^{-1}A$-module (called the ring of fractions of M with denominators in S);

for every $s \in S$, $s/1$ is invertible in $S^{-1}A$; its inverse is $1/s$. The canonical mapping i_A^S (resp i_M^S) is a homomorphism of rings (resp. a homomorphism of A modules, $S^{-1}M$ being considered an A module via the homomorphism $i_A^S : A \to S^{-1}A$).

(1.2.3) If $S_f = {f^n}_{n \geq 0}$ for each $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered an algebra over A, we write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT-1)A[T]$. When $f = 1$, A_f and M_f are canonically identical to A and M; if f is nilpotent, A_f and M_f reduce to 0.

(1.2.4) the ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are the solution to a universal problem: for every homomorphism u from A to a ring B such that u(S) is made up of invertible elements in B, there is a factorization of the form

$$u : A \xrightarrow[i_A^S]{} S^{-1}A \xrightarrow[u^*]{} B$$

where u^* is a homomorphism of rings. Under the same hypotheses, let M be a A-module, N a B-module, $v : m \to N$ a homomorphism of A modules (with the structure of an A module on N defined by $u : A \to B$); then v has a factorization of the form

$$v : M \xrightarrow[i_M^S]{} S^{-1}M \xrightarrow[v^*]{} N$$

where v^* is a homomorphism of $S^{-1}A$ modules (with the structure of an $S^{-1}A$ module on N given by u^*)

(1.2.5) One defines a canonical isomorphism $S^{-1}A \bigotimes_A M \simeq S^{-1}M$ of $S^{-1}A$ modules, which takes an element $(a/s) \otimes m$ to the element $(am)/s$, with inverse mapping taking m/s to $(1/s) \otimes m$.

(1.2.6) For each ideal a' in $S^{-1}A$, $a = (i_A^S)^{-1}(a')$ is an ideal in A, and a' is the ideal in $S^{-1}A$ generated by $i_A^S(a)$, which is denoted $S^{-1}a$ (1.3.2). The mapping $p' \to (i_A^S)^{-1}(p')$ is an isomorphism of ordered sets between the set of prime ideals in $S^{-1}A$ and the set of prime ideals in A such that $p \cap S = \emptyset$. Furthermore, the local rings A_p and $(S^{-1}A)_{S^{-1}p}$ are canonically isomorphic (1.5.1).

(1.2.7) When A is an integral domain, and K is its field of fractions, the canonical mapping $i_A^S : A \to S^{-1}A$ is injective for all multiplicative subsets S which do not contain 0, and $S^{-1}A$ is canonically identified with a subring of K containing A. In particular, for every prime ideal p of A, A_p is a local ring containing A, with maximal ideal pA_p, and one has $pA_p \cap A = p$.

(1.2.8) If A is reduced (1.1.1) then the same is true of $S^{-1}A$: in fact, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, there exists $s' \in S$ such than $s'x^n = 0$, or $(s'x)^n = 0$, which under the hypothesis implies $s'x = 0$, so $x/s = 0$.

0.1 1.3. Functorial properties

(1.3.1) Let M, N be two A-modules, u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, one defines a $S^{-1}A$ homomorphism $S^{-1}M \to S^{-1}N$, denoted $S^{-1}u$, by $(S^{-1}u)(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are identified with $S^{-1}A \bigotimes_A M$ and $S^{-1}A \bigotimes_A N$ (1.2.5), $S^{-1}u$ is identified with $1 \otimes u$. If P is a third A module, and v is an A-homomorphism $N \to P$, one has $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; so we say that $S^{-1}M$ is a contravariant functor in M, from the category of A modules to that of $S^{-1}A$ modules (where A and S are fixed).

(1.3.2) The functor $S^{-1}M$ is exact; in other words, if the sequence

$$M \xrightarrow[u]{} N \xrightarrow[v]{} P$$

is exact, the same is true of the sequence

$$S^{-1}M \xrightarrow[S^{-1}u]{} S^{-1}N \xrightarrow[S^{-1}v]{} S^{-1}P$$

. In particular, if $u : m \to N$ is injective (resp. surjective), the same is true of $S^{-1}u$; if N and P are two submodules of M, $S^{-1}N$ and $S^{-1}P$ are canonically identified as submodules of $S^{-1}M$, and one has

$$S^{-1}(N + P) = S^{-1}N + S^{-1}P \text{ and } S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P)$$

.

(1.3.3) Let $(M_\alpha, \phi_{\beta\alpha})$ be an inductive system of A modules; then $(S^{-1}M_\alpha, S^{-1}\phi_{\beta\alpha})$ is a inductive system of $S^{-1}A$ modules. Expressing $S^{-1}M$ and $S^{-1}\phi_{\beta\alpha}$ as tensor products (1.2.5 and 1.3.1) the result on the commutativity of the operations of tensor product and inductive limit gives a canonical isomorphism

$$S^{-1}lim(M_\alpha) \simeq lim(S^{-1}M_\alpha)$$

which we express by saying that the functor $S^{-1}M$ (in M) commutes with inductive limits.

(1.3.4) Let M, N be two A-modules; there exists a canonical functorial isomorphism

$$S^{-1}M \bigotimes_{S^{-1}A} (S^{-1}N) \simeq S^{-1}(M \bigotimes_A N)$$

which takes $(m/s) \otimes (n/t)$ to $(m \otimes n)/st$.

(1.3.5) One similarly has a natural transformation (in M and N)

$$S^{-1}\text{Hom}_A(M,N) \to \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which, given u/s, maps it to to $m/t \to u(m)/st$. If M is finitely presented, this transformation is an equivalence; this is immediate if M is of the form A^r, and one proves it in general for a sequence $A^p \to A^q \to M \to 0$ by using the exactness of the functor $S^{-1}M$ and the left exactness of $\text{Hom}_A(M,N)$ in M. One notes that this is always the case when A is Notherian, and the A module M is of finite type.

1.4. Change of multiplicative subsets

(1.4.1) Let S and T be two multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) from $S^{-1}A$ to $T^{-1}A$, which takes an element denoted a/s in $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; one has $i_A^T = \rho_A^{T,S} \circ i_A^S$. For each A module M, there similarly exists an $S^{-1}A$ linear mapping from $S^{-1}M$ to $T^{-1}M$ (where the latter is considered a $S^{-1}A$ module via the homomorphism $\rho_A^{T,S}$), which takes the element m/s in $S^{-1}M$ to the element m/s in $T^{-1}M$; one denotes this mapping $\rho_M^{T,S}$, or simply $rho^{T,S}$, and one again has $i_M^T = \rho_M^{T,S} \circ i_M^S$; under the canonical identification (1.2.5), $\rho_A^{T,S}$ is identified with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a natural transformation from the functor $S^{-1}M$ to the functor $T^{-1}M$; in other words, the diagram

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}u} & S^{-1}N \\ {\scriptstyle\rho_M^{T,S}}\downarrow & & \downarrow{\scriptstyle\rho_N^{T,S}} \\ T^{-1}M & \xrightarrow{T^{-1}u} & T^{-1}N \end{array}$$

is commutative for each homomorphism $u : M \to N$; one further notes that $T^{-1}u$ is entirely determined by $S^{-1}u$, because if $m \in M$ and $t \in T$, one has

$$(T^{-1}u)(m/t) = (t/1)^{-1}\rho^{T,S}((S^{-1}u)(m/1))$$

.

(1.4.2) With the same notation, for two A modules M,N the diagrams (cf 1.3.4 and 1.3.5)

$$\begin{array}{ccc} (S^{-1}M)\bigotimes_{S^{-1}A}(S^{-1}N) & \xrightarrow{\simeq} & S^{-1}(M\bigotimes_A N) \\ \downarrow & & \downarrow \\ (T^{-1}M)\bigotimes_{T^{-1}A}(T^{-1}N) & \xrightarrow{\simeq} & T^{-1}(M\bigotimes_A N) \end{array}$$

and

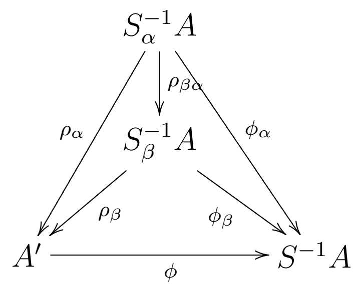
$$\begin{array}{ccc} S^{-1}\mathrm{Hom}_A(M,N) & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \\ \downarrow & & \downarrow \\ T^{-1}\mathrm{Hom}_A(M,N) & \longrightarrow & \mathrm{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{array}$$

are commutative.

(1.4.3) In the important case in which when $\rho^{T,S}$ is bijective, one knows that every element of T is a divisor of an element in S; one thus identifies via $\rho^{T,S}$ the modules $S^{-1}M$ and $T^{-1}M$. One says that S is saturated if every divisor in A of an element of S is in S; one can replace S by a set T which contains all the divisors of elements of S (a set which is multiplicative and saturated), and one can limit consideration to modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4) If S, T, U are three multiplicative subsets of A, such that $S \subset T \subset U$, one has $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

(1.4.5) Consider an finite filtered family of multiplicative subsets (S_α) of A (one writes $\alpha \leq \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\cup_\alpha S_\alpha$; we put $\rho_{\beta\alpha} = \rho_A^{S_\beta, S_\alpha}$ for $\alpha \leq \beta$; by virtue of (1.4.4) the homomorphism $\rho_\beta \alpha$ defines a ring A' as the inductive limit of the system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let ρ_α be the canonical mapping $S^{-1}A \to A'$, and set $\phi_\alpha = \rho_A^{S,S_\alpha}$; as $\phi_\alpha = \phi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leq \beta$ by (1.4.4) one defines a unique homomorphism $\phi : A' \to S^{-1}A$ by the commutative diagram



In fact, ϕ is an isomorphism. It is immediate from the construction that ϕ is surjective. For injectivity, if $\rho_\alpha(a/s_\alpha) \in A'$ is such that $\phi(\rho_\alpha(a/s_\alpha)) = 0$, this implies that $a/s_\alpha = 0$ in $S^{-1}A$, and so there exists $s \in S$ such that sa=0; but there is the $\beta \geq \alpha$ such that $s \in S_\beta$, and it follows that $\rho_\alpha(a/s_\alpha) = \rho_\beta(sa/ss_\alpha) = 0$, and ϕ is injective. One treats similarly the cas of an A module M, and one defines canonical isomorphisms

$$\lim S_\alpha^{-1}A \simeq (\lim S_\alpha)^{-1}A, \lim S_\alpha^{-1}M \simeq (\lim S_\alpha)^{-1}M$$

the second being functorial in M.

(1.4.6) Let S_1 and S_2 be two multiplicative subsets of A, then S_1S_2 is also a multiplicative subset of A. Designate by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For each A module M, there exists a functorial isomorphism

$$(S_2'^{-1})S_1^{-1}M) \simeq (S_1S_2)^{-1}M$$

which takes $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of Rings

(1.5.1) let A, A' be two rings, ϕ a homomorphism $A' \to A$, S (resp S') a multiplicative subset of A(resp A) such that $\phi(S') \subset S$; the composition $A' \to A \to S^{-1}A$ factorizes as $A' \to S'^{-1}A' \to S^{-1}A$ by virtue of (1.2.4); one has $\phi^{S'}(a'/s') = \phi(a')/\phi(s')$. if $A = \phi(A')$ and $S = \phi(S')$, $\phi^{S'}$ is surjective. If $A' = A$ and ϕ is the identity, $\phi^{S'}$ is none other than the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1)

(1.5.2) Under the hypotheses of (1.5.1) let M be an A module. There exists a canonical natural transformation $\sigma : S'^{-1}(M_{[\phi]}) \to (S^{-1}M)_{[\phi^{S'}]}$ of $S'^{-1}A'$ modules, that corresponds each element m/s' of $S'^{-1}(M_{[\phi]})$, to the element $m/\phi(s')$ of $(S^{-1}M)_{[\phi^{S'}]}$; one verifies immediately from the definitions that this does not depend on the expression m/s' of the element considered. When $S = \phi(S')$, the homomorphism σ is bijective. When $A' = A$ and ϕ is the identity, σ, is none other than the homomorphism $\rho_M^{S,S'}$ defined in 1.4.1. When in particular $M = A$, the homomorphism ϕ defines in A the structure of an A' algebra; $S'^{-1}(A_{[\phi]})$ has then the structure of a ring, identified with $(\phi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\phi]}) \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$ algebras.

(1.5.3) Let M and N be two A modules; one composes the homomorphisms defined in (1.3.4) and (1.5.2) to get a homomorphism

$$(S^{-1}M \bigotimes_{S^{-1}A} S^{-1}N)_{[\phi^{S'}]} \leftarrow S'^{-1}((M \bigotimes_A N)_{[\phi]})$$

which is an isomorphism when $\phi(S') = S$. Similarly, one composes the homomorphisms (1.3.5) and (1.5.2) to get the homomorphism

$$S'^{-1}((Hom_A(M, N))_{[\phi]}) \to (Hom_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\phi^{S'}]}$$

which is an isomorphism when $\phi(S') = S$ and M admits a finite presentation.

(1.5.4) We consider an A' module N', formed as the tensor product $N' \bigotimes_{A'} A_{[\phi]}$, which can be considered as an A module via a.$(n \otimes b) = n' \otimes (ab)$. There exist an functorial isomorphism of $S^{-1}A$ modules

$$\tau : (S'^{-1}N') \bigotimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} \simeq S^{-1}(N' \bigotimes_{A'} A_{[\phi]})$$

which, for an element $(n'/s') \otimes (a/s)$, gives the element $(n' \otimes a)/(\phi(s')s)$; one seperately verifies the effect of replacing n'/s' (resp a/s) by a different expression of the same element $(n' \otimes a)/(\phi(s')s)$ is no change; for the other part, one defines an inverse for τ by mapping $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: use the fact that $S^{-1}(N' \bigotimes_{A'} A_{[\phi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\phi]}) \otimes_A S^{-1}A$ (1.2.5) therefore also to $N' \otimes_{A'} (S^{-1}A)_{[\psi]}$, where ψ is the homomorphism $a' \to \phi(a')/1$ from A' to $S^{-1}A$.

(1.5.5) If M' and N' are two A' modules, one composes the isomorphisms (1.3.4) and (1.5.4) to obtain the isomorphism

$$S'^{-1}M' \bigotimes_{S'^{-1}A'} S'^{-1}N' \bigotimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(M' \bigotimes_{A'} N' \bigotimes_{A'} A)$$

similarly, if M' admits a finite presentation, one composes (1.3.5) and (1.5.4) to get an isomorphism

$$\mathrm{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \bigotimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(\mathrm{Hom}_{A'}(M', N') \bigotimes_{A'} A).$$

(1.5.6) Under the hypotheses of (1.5.1) let T (resp T') be a second multiplicative subset of A (resp A') such that $S \subset T$ (resp $S' \subset T'$) and $\phi(T') \subset T$. Then the diagram

$$\begin{array}{ccc} S'^{-1}A' & \xrightarrow{\phi^{S'}} & S^{-1}A \\ \downarrow{\scriptstyle \rho^{T',S'}} & & \downarrow{\scriptstyle \rho^{T,S}} \\ T'^{-1}A' & \xrightarrow[\phi^{T'}]{} & T^{-1}A \end{array}$$

is commutative. If M is an A module, the diagram

$$\begin{array}{ccc} S'^{-1}M'_{[\phi]} & \xrightarrow{\sigma} & (S^{-1}M)_{[\phi^{S'}]} \\ \downarrow{\scriptstyle \rho^{T',S'}} & & \downarrow{\scriptstyle \rho^{T,S}} \\ T'^{-1}M_{[\phi]} & \xrightarrow[\sigma]{} & (T^{-1}M)_{[\phi^{T'}]} \end{array}$$

is commutative. Finally, if N' is an A' module, the diagram

$$\begin{array}{ccc} (S'^{-1}N') \bigotimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} & \xrightarrow[\tau]{\simeq} & S^{-1}(N' \bigotimes_{A'} A_{[\phi]}) \\ \downarrow & & \downarrow{\scriptstyle \rho^{T,S}} \\ (T'^{-1}N') \bigotimes_{T'^{-1}A'} (T^{-1}A)_{[\phi^{T'}]} & \xrightarrow[\simeq]{\tau} & T^{-1}(N' \bigotimes_{A'} A_{[\phi]}) \end{array}$$

is commutative, the vertical arrow on the left being obtained by applying $\rho_{N'}^{T',S'}$ to $S'^{-1}N'$ and $\rho_A^{T,S}$ to $S^{-1}A$.

(1.5.7) Let A'' be a third ring, $\phi' : A'' \to A'$ a homomorphism of rings, S'' a multiplicative subset of A'' such that $\phi'(S'') \subset S'$. Put $\phi'' = \phi \circ \phi'$; then one has

$$\phi''^{S''} = \phi^{S'} \circ \phi'^{S''}.$$

Let M be an A module; it is evident that $M_{[\phi'']} = (M_{[\phi]})_{[\phi']}$; if σ' and σ'' are the homomorphisms defined by ϕ' and ϕ'' as σ is defined by ϕ as in (1.5.2), one has the formula of transitivity

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N" be a A"-module; the A module $N'' \otimes_{A'} A_{[\phi'']}$ is canonically identified with $(N'' \otimes_{A''} A'_{[\phi']}) \otimes_{A'} A_{[\phi]}$, and similarly, the $S^{-1}A$ module $(S''^{-1}N'') \bigotimes_{S''^{-1}A''} (S^{-1}A)_{[\phi''^{S''}]}$ is canonically identified with

$$((S''^{-1}N'') \bigotimes_{S''^{-1}A''} (S'^{-1}A')_{[\phi'^{S'}]}) \bigotimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]}$$

. With these identifications, if τ' and τ'' are the isomorphisms defined with respect to ϕ' and ϕ'' and τ is defined as in (1.5.4) with respect to ϕ, one has the formula of transitivity

$$\tau'' = \tau \circ (\tau' \otimes 1)$$

(1.5.8) Let A be a subring of a ring B; for every minimal prime ideal $\mathfrak{p}$ of A, there exists a minimal prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. In fact, $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$ (1.3.2) and posseses a single prime ideal $\mathfrak{p}'$ (1.2.6); if $B_{\mathfrak{p}}$ does not reduce to zero, it too posseses a prime ideal $\mathfrak{q}'$ and necessarily $\mathfrak{q}' \cap A_{\mathfrak{p}} = \mathfrak{p}'$; the prime ideal $\mathfrak{q}_1$ in B is the inverse image of $\mathfrak{q}'$ and is therefore such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and a fortiori one has $\mathfrak{q} \cap A = \mathfrak{p}$ for each prime ideal $\mathfrak{q}$ of B containing $\mathfrak{q}_1$.

1.6. Identification of a module M_f as an inductive limit.

(1.6.1) Let M be an A module, f an element of A. Consider the sequence (M_n) of A modules, all identically M, such that for each pair of integers

$m \leq n$, ϕ_{nm} is the homomorphism z$\rightarrow f^{n-m}z$ from M_m to M_n; it is immediate than $((M_n),(\phi_{nm}))$ is an inductive system of A modules; let $N = \lim M_n$ be the inductive limit of the system. We define a canonical A-isomorphism from N to M_f. For this, we remark that, for all n, $\theta_n : z \rightarrow z/f^n$ is an A-homomorphism from $M = M_i$ to M_f, and it results from the definitions that one has $\theta_n \circ \phi_{nm} = \theta_m$ for $m \leq n$. There therefore exists an A-homomorphism $\theta : N \rightarrow M_f$ such that, if ϕ_n denotes the canonical homomorphism $M_n \rightarrow N$, one has $\theta_n = \theta \circ \phi_n$. With this holding, every element of M_f is of the form z/f^n for some n, so it is clear that ϕ is surjective. For the other part, if $\theta(\phi_n(z)) = 0$, or in other words, $z/f^n = 0$, there exist an integer $k > 0$ such that $f^k z = 0$, so $\phi_{n+k,n}(z) = 0$, which implies $\phi_n(z) = 0$. One identifies M_f and $\lim M_n$ via θ.

(1.6.2) We use the notation $M_{f,n}$, ϕ^f_{nm} and ϕ^f_n instead of M_n, ϕ_{nm} and ϕ_n. Let g be a second element of A. Since f^n divides $f^n g^n$, one has a functorial homomorphism $\rho_{fg,f} : M_f \rightarrow M_{fg}$ (1.4.1 and 1.4.3); if one identifies M_f and M_{fg} with $\lim M_{f,n}$ and $\lim M_{fg,n}$ respectively, $\rho_{fg,f}$ is identified with the inductive limit of maps $\rho^n_{fg,f} : M_f, n \rightarrow M_{fg,n}$ defined by $\rho^n_{fg,f}(z) = g^n z$. In fact, this result is immediate from the commutativity of the diagram

$$\begin{array}{ccc} M_{f,n} & \xrightarrow{\rho^n_{fg,n}} & M_{fg,n} \\ \downarrow{\scriptstyle \phi^f_n} & & \downarrow{\scriptstyle \phi^{fg}_n} \\ M_f & \xrightarrow{\rho_{fg,f}} & M_{fg} \end{array}$$

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1.7. Support of a module

(1.7.1) Given an A module M, we name the support of M and denote by Supp(M) the set of prime ideals $\mathfrak{p}$ in A such that $M_\mathfrak{p} \neq 0$. For M to equal 0, it is necessary and sufficient that Supp(M)=$\emptyset$ for if $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$, the annihilator of an element $x \in M$ is not contained in any prime ideal of A, hence is invertible in A.

(1.7.2) If $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ is an exact sequence of A modules, one has Supp(M)=Supp(N)$\cup$ Supp(P) since for all prime ideals in A, the sequence $0 \rightarrow N_\mathfrak{p} \rightarrow M_\mathfrak{p} \rightarrow P_\mathfrak{p} - > 0$ is exact (1.3.2) and $M_\mathfrak{p} = 0$ if and only if $N_\mathfrak{p} = P_\mathfrak{p} = 0$.

(1.7.3) If M is the sum of a family (M_λ) of submodules, $M_\mathfrak{p}$ is the sum of $(M_\lambda)_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ in A (1.3.3 and 1.3.2) so

$$\mathrm{Supp}(M) = \cup_\lambda \mathrm{Supp}(M_\lambda)$$

(1.7.4) If M is an A module of finite type, Supp(M) is the set of prime ideals which contain the annihilator of M. In fact, if M is generated by a single element x, then $M_\mathfrak{p} = 0$ implies there exist $s \notin \mathfrak{p}$ such that $s.x = 0$, so that $\mathfrak{p}$ is not contained in the annihilator of x. If M admits a finite system $(x_i)_{1 \leq i \leq n}$ of generators and if a_i is the annihilator of x_i, by (1.7.3) Supp(M) is the set of $\mathfrak{p}$ containing the a_i, where, by the same, the set $\mathfrak{p}$ contains $a = \cap_i a_i$, which is the annihilator of M.

(1.7.5) If M and N are two A modules of finite type, one has

$$\mathrm{Supp}(M \bigotimes_A N) = \mathrm{Supp}(M) \cap Supp(N)$$

One can see that if $\mathfrak{p}$ is a prime ideal of A, the condition $M_p \bigotimes_{A_p} N_p \neq 0$ is equivalent to $<<$ $M_p \neq 0$ and $N_p \neq 0$ $>>$) (compare (1.3.4)). In other words, one sees that if P,Q are two modules of finite type over a local ring B which does not reduce to zero, then $P \otimes_B Q \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of B. By Nakayama's lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ do not reduce to 0, hence the same is true of the tensor products $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}B} (Q/\mathfrak{m}Q) = (P \bigotimes_B Q) \otimes_B (B/\mathfrak{m})$, thus the conclusion. In particular, if M is an A module of finite type, $\mathfrak{a}$ an ideal of A, $\mathrm{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing $\mathfrak{a}$ as well as the annihilator $\mathfrak{n}$ of M (1.7.4), or the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.