If the category $\mathbf{K}$ admits projective limits (in the general sense), and if $\mathcal{B}, \mathcal{B}'$ are bases for the topologies of X and Y respectively, then to define a ψ-morphism u of sheaves, it suffices to give $u_{U,V}$ for $U \in \mathcal{B}$, $V \in \mathcal{B}'$, and $\psi(U) \subseteq V$, checking compatibility conditions (3.5.1.1) for $U, U' \in \mathcal{B}$, $V, V' \in \mathcal{B}'$. Define, then, u_W, for any open $W \subseteq Y$, to be any projective limit of $u_{U,V}$ where $V \in \mathcal{B}'$, $V \subseteq W$ and $U \in \mathcal{B}$, $\psi(U) \subseteq V$.

When the category $\mathbf{K}$ admits inductive limits, one has, for all $x \in X$, a morphism $\mathcal{G}(V) \to \mathcal{F}(\psi^{-1}(V)) \to \mathcal{F}_x$ for every open neighborhood V of $\psi(x)$ in Y. These morphisms form an inductive system which yields a morphism $\mathcal{G}_{\psi(x)} \to \mathcal{F}_x$ by passing to the limit.

(3.5.2) Under the hypotheses of (3.4.3), let $\mathcal{F}, \mathcal{G}, \mathcal{K}$ be $\mathbf{K}$-valued presheaves of X, Y, Z respectively, and let $u : \mathcal{G} \to \psi_*(\mathcal{F})$, $v : \mathcal{K} \to \psi_*(\mathcal{G})$ be a ψ-morphism and a ψ'-morphism respectively. One can derive a ψ''-morphism $w : \mathcal{K} \xrightarrow{v} \psi'_*(\mathcal{G}) \xrightarrow{\psi'_*(u)} \psi'_*(\psi_*(\mathcal{F})) = \psi''_*(\mathcal{F})$, called the composition of u and v. One can thus consider couples $(X, \mathcal{F})$ consisting of a topological space X and a presheaf $\mathcal{F}$ on X (with values in $\mathbf{K}$) to form a category, the morphisms being couples $(\psi, \theta) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ consisting of a continuous map $\psi : X \to Y$ and a ψ-morphism $\theta : \mathcal{G} \to \mathcal{F}$.

(3.5.3) Let $\psi : X \to Y$ be a continuous map, $\mathcal{G}$ a presheaf on Y (with values in $\mathbf{K}$). We define the *inverse image of* $\mathcal{G}$ *by* ψ to be a couple $(\mathcal{G}', \rho)$, where $\mathcal{G}'$ is a $\mathbf{K}$-valued sheaf on X and $\rho : \mathcal{G} \to \mathcal{G}'$ is a ψ-morphism (ie-a morphism $\mathcal{G} \to \psi_*(\mathcal{G}')$) such that, for every $\mathbf{K}$-valued sheaf $\mathcal{F}$ of X, the map

(3.5.3.1) $Hom_X(\mathcal{G}', \mathcal{F}) \to Hom_\psi(\mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, \psi_*(\mathcal{F}))$

taking v to $\psi_*(v) \circ \rho$, is a bijection. This map, being functorial in $\mathcal{F}$, defines an isomorphism of functors in $\mathcal{F}$. The couple $(\mathcal{G}', \rho)$, being universal, is determined up to unique isomorphism when it exists. We then write $\mathcal{G}' = \psi^*(\mathcal{G})$, $\rho = \rho_\mathcal{G}$, and by abuse of language, we call $\psi^*(\mathcal{G})$ the *inverse image sheaf* of $\mathcal{G}$ by ψ, which is equipped with a canonical ψ-morphism $\rho_\mathcal{G} : \mathcal{G} \to \psi^*(\mathcal{G})$ (ie-the canonical morphism

(3.5.3.2) $\rho_\mathcal{G} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$

of presheaves on Y).

For every homomorphism $v : \psi^*(\mathcal{G}) \to \mathcal{F}$ (where $\mathcal{F}$ is a $\mathbf{K}$-valued sheaf on X), we have $v^\flat = \psi_*(v) \circ \rho_\mathcal{G}$. By definition, every morphism of presheaves $u : \mathcal{G} \to \psi_*(\mathcal{F})$ is of the form $v^\flat$ for one and only one v, which we denote $u^\sharp$. In other words, every morphism of presheaves $u : \mathcal{G} \to \psi_*(\mathcal{F})$ factors uniquely as

(3.5.3.3) $u : \mathcal{G} \xrightarrow{\rho_\mathcal{G}} \psi_*(\psi^*(\mathcal{G})) \xrightarrow{\psi_*(u^\sharp)} \psi_*(\mathcal{F})$.

(3.5.4) Suppose that the category **K** is such that every presheaf $\mathcal{G}$ on Y with values in **K** has an inverse image by ψ, denoted $\psi^*(\mathcal{G})$ [1].

We will see that one can define $\psi^*(\mathcal{G})$ to be a covariant functor on $\mathcal{G}$ from the category of **K**-valued presheaves on Y to the category of **K**-valued presheaves on X, such that the isomorphism $v \to v^\flat$ is an isomorphism of bifunctors

(3.5.4.1) $Hom_X(\psi^*(\mathcal{G}), \mathcal{F}) \xrightarrow{\sim} Hom_Y(\mathcal{G}, \psi_*(\mathcal{F}))$

on $\mathcal{G}$ and $\mathcal{F}$.

Indeed, for every morphism $w : \mathcal{G}_1 \to \mathcal{G}_2$ of **K**-valued presheaves on Y, we consider the composition morphism $\mathcal{G}_1 \xrightarrow{w} \mathcal{G}_2 \xrightarrow{\rho_{\mathcal{G}_2}} \psi_*(\psi^*(\mathcal{G}_2))$. There corresponds to this a morphism $(\rho_{\mathcal{G}_2} \circ w)^\sharp : \psi^*(\mathcal{G}_1) \to \psi^*(\mathcal{G}_2)$, which we denote $\psi^*(w)$. One now has, under the conditions of (3.5.3.3)

(3.5.4.2) $\psi_*(\psi^*(w)) \circ \rho_{\mathcal{G}_1} = \rho_{\mathcal{G}_2} \circ w$.

For every morphism $u : \mathcal{G}_2 \to \psi_*(\mathcal{F})$ where $\mathcal{F}$ is a **K**-valued sheaf on X, one has, via (3.5.3.3), (3.5.4.2), and the definition of $u^\flat$:

$$(u^\sharp \circ \psi^*(w))^\flat = \psi_*(u)^\sharp \circ \psi_*(\psi^*(w)) \circ \rho_{\mathcal{G}_1} = \psi_*(u^\sharp) \circ \rho_{\mathcal{G}_2} \circ w = u \circ w$$

or

(3.5.4.3) $(u \circ w)^\sharp = u^\sharp \circ \psi^*(w)$.

If, in particular, we take u to be the morphism $\mathcal{G}_2 \xrightarrow{w'} \mathcal{G}_3 \xrightarrow{\rho_{\mathcal{G}_3}} \psi_*(\psi^*(\mathcal{G}_3))$, we have $\psi^*(w' \circ w) = (\rho_{\mathcal{G}_3} \circ w' \circ w)^\sharp = (\rho_{\mathcal{G}_3} \circ w)^\sharp \circ \psi^*(w) = \psi^*(w') \circ \psi^*(w)$, which proves our assertion.

Finally, for every **K**-valued sheaf $\mathcal{F}$ on X, let $i_{\mathcal{F}}$ be the identity morphism on $\psi_*(\mathcal{F})$ and denote by $\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}$ the morphism $(i_{\mathcal{F}})^\sharp$. The formula (3.5.4.3) gives, in particular, the factorization

(3.5.4.4) $u^\sharp : \psi^*(\mathcal{G}) \xrightarrow{\psi^*(u)} \psi^*(\psi_*(\mathcal{F})) \xrightarrow{\sigma_{\mathcal{F}}} \mathcal{F}$

for every morphism $u : \mathcal{G} \to \psi_*(\mathcal{F})$. We do not claim that the morphism $\sigma_{\mathcal{F}}$ is canonical.

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[1] In a book cited in the introduction we give three general conditions on the category **K** which ensure the existence of inverse images of **K**-valued presheaves.

(3.5.5) Let $\psi' : Y \to Z$ be a continuous map, and suppose that every **K**-valued presheaf $\mathcal{H}$ of Z admits an inverse image $\psi'^*(\mathcal{H})$ by ψ'. Then (under the hypotheses of (3.5.4)) all **K**-valued presheaves $\mathcal{H}$ of Z admit an inverse image by $\psi'' = \psi' \circ \psi$ and there is a canonical functorial isomorphism

$$(3.5.5.1)\ \psi''^*(\mathcal{H}) \xrightarrow{\sim} \psi^*(\psi'^*(\mathcal{H})).$$

This is, indeed, immediate from the definitions, taking into account that $\psi'' = \psi' \circ \psi$. Moreover, if $u : \mathcal{G} \to \psi_*(\mathcal{F})$ is a ψ-morphism, $v : \mathcal{H} \to \psi'_*(\mathcal{G})$ a ψ'-morphism, and $w = \psi'_*(u) \circ v$ their composition (3.5.2), we note immediately that $w^\sharp$ is the composition morphism

$$w^\sharp : \psi^*(\psi'^*(\mathcal{H})) \xrightarrow{\psi^*(v^\sharp)} \psi^*\mathcal{G} \xrightarrow{u^\sharp} \mathcal{F}\ .$$

(3.5.6) In particular, take ψ to be the identity map $id_X : X \to X$. If the inverse image by ψ of a **K**-valued presheaf $\mathcal{F}$ on X exists, it is the inverse image of the sheaf associated to the presheaf $\mathcal{F}$. Every morphism $u : \mathcal{F} \to \mathcal{F}'$ from $\mathcal{F}$ to a **K**-valued sheaf $\mathcal{F}'$ factors uniquely as $\mathcal{F} \xrightarrow{\rho_{\mathcal{F}}} id_X^*(\mathcal{F}) \xrightarrow{u^\sharp} \mathcal{F}'$.

3.6. Simple Sheaves and Locally Simple Sheaves

3.6.1 We will say that a **K**-valued presheaf $\mathcal{F}$ of X is *constant* if the canonical morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism for every nonempty open $U \subseteq X$. One notes that $\mathcal{F}$ is not necessarily a sheaf. A sheaf is said to be *simple* if it is associated (3.5.6) to a constant presheaf. A sheaf $\mathcal{F}$ is said to be *locally simple* if every $x \in X$ has an open neighborhood U such that $\mathcal{F}_{|U}$ is simple.

(3.6.2) Suppose that X is irreducible (2.1.1); then the following are equivalent:

a $\mathcal{F}$ is a constant presheaf on X.

b $\mathcal{F}$ is a simple sheaf on X.

c $\mathcal{F}$ is a locally simple sheaf on X.

Indeed, let $\mathcal{F}$ be a constant presheaf on X. If U, V are nonempty open subsets of X and $U \cap V$ is nonempty, then, as $\mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}(U \cap V)$ and $\mathcal{F}(X) \to \mathcal{F}(U)$ are isomorphisms, $\mathcal{F}(U) \to \mathcal{F}(U \cap V)$ and $\mathcal{F}(V) \to \mathcal{F}(U \cap V)$ are also isomorphisms. One immediately concludes that axiom (F) of (3.1.2) holds, $\mathcal{F}$ is isomorphic to its associated sheaf, and a) implies b).

Now let (U_α) be a cover of X by nonempty open sets and let $\mathcal{F}$ be a sheaf on X such that $\mathcal{F}_{|U_\alpha}$ is simple for all α. Since U_α is irreducible, $\mathcal{F}_{U_\alpha}$ is a constant presheaf on X by the preceding paragraph. As $U_\alpha \cap U_\beta$ is nonempty, $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\alpha \cap U_\beta)$ and $\mathcal{F}(U_\beta) \to \mathcal{F}(U_\alpha \cap U_\beta)$ are isomorphisms from which one can derive a canonical isomorphism $\theta_{\alpha\beta} : \mathcal{F}(U_\alpha) \to \mathcal{F}(U_\beta)$ for each pair of indices. But then, applying condition (F) for $U = X$, we see that for every index α_0, $\mathcal{F}(U_{\alpha_0})$ and $\theta_{\alpha_0\alpha}$ are universal, which (by unicity) implies that $\mathcal{F}(X) \to \mathcal{F}(U_{\alpha_0})$ is an isomorphism, showing c) implies a).

Inverse Images of Presheaves of Groups or Rings

(3.7.1) We will show that when one takes **K** to be the category of sets, the inverse image by ψ of every **K**-valued presheaf $\mathcal{G}$ must exist (the notation and hypotheses for X, Y, ψ being those of (3.5.3)). Indeed, for every open $U \subseteq X$, $\mathcal{G}'(U)$ is defined as follows: an element s' of $\mathcal{G}'(U)$ is a family $(s'_x)_{x\in U}$ where $s'_x \in \mathcal{G}_{\psi(x)}$ for every $x \in U$ and where, for every $x \in U$, the following condition is satisfied: there exists an open neighborhood V of $\psi(x)$ in Y, a neighborhood $W \subseteq \psi^{-1}(V) \cap U$ of x, and an element $s \in \mathcal{G}(V)$ such that $s'_z = s_{\psi(z)}$ for every $z \in W$. One verifies immediately that $U \mapsto \mathcal{G}'(U)$ satisfies the sheaf axioms.

Let $\mathcal{F}$ be a sheaf of sets on X with morphisms $u : \mathcal{G} \to \psi_*(\mathcal{F})$, $v : \mathcal{G}' \to \psi_*(\mathcal{F})$. One defines $u^\sharp$ and $v^\flat$ in the following way: if s' is a section of $\mathcal{G}'$ over a neighborhood U of $x \in X$, V is an open neighborhood of $\psi(x)$, and $s \in \mathcal{G}(V)$ satisfies $s'_z = s_{\psi(z)}$ for all z in some neighborhood of x contained in $\psi^{-1}(V) \cap U$, then one defines $u^\sharp_x(s'_x) = u_{\psi(x)}(s_{\psi(x)})$. Similarly, if $s \in \mathcal{G}(V)$ (V open in Y), $v^\flat(s)$ is the section of $\mathcal{F}$ over $\psi^{-1}(V)$ whose image under v is the section s' of $\mathcal{G}'$ such that $s'_x = s_{\psi(x)}$ for every $x \in \psi^{-1}(V)$. Moreover, the canonical homomorphism (3.5.3) $\rho : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$ is defined in the following way: for every open $V \subseteq Y$ and every section $s \in \Gamma(V, \mathcal{G})$, $\rho(s)$ is the section $(s_{\psi(x)})_{x\in\psi^{-1}(V)}$ of $\psi^*(\mathcal{G})$ over $\psi^{-1}(V)$. The verification of the relations $(u^\sharp)^\flat = u$, $(v^\flat)^\sharp = v$, and $v^\flat = \psi_*(v) \circ \rho$ is immediate, proving our assertion.

One checks that, if $w : \mathcal{G}_1 \to \mathcal{G}_2$ is a homomorphism of presheaves of sets on Y, $\psi^*(w)$ may be made explicit in the following way: if $s' = (s'_x)_{x\in U}$ is a section of $\psi^*(\mathcal{G}_1)$ over an open subset U of X, $(\psi^*(w))(s')$ is the family $(w_{\psi(x)}(s'_x))_{x\in U}$. Finally, it is immediate that, for every open V of Y, the inverse image of $\mathcal{G}_{|V}$ by the restriction of ψ to $\psi^{-1}(V)$ is equal to the sheaf induced by $\psi^*(G)_{|\psi^{-1}(V)}$.

When ψ is the identity id_X, the definition coincides with that of a sheaf of sets associated to a presheaf (G, II, 1.2). The preceding considerations apply without change when **K** is the category of groups or (not necessarily commutative) rings.

When X is an arbitrary subset of a topological space Y and j is the canonical inclusion $X \to Y$, for all **K**-valued sheaves $\mathcal{G}$ on Y, we define the *sheaf on X induced by $\mathcal{G}$* to be the inverse image $j^*(\mathcal{G})$ (when it exists); for sheaves of sets (or groups or rings) this coincides with the usual definition.

(3.7.2) Using the notation and hypotheses of (3.5.3), let $\mathcal{G}$ be a sheaf of groups [resp. rings] on Y. The definition of sections of $\psi^*(\mathcal{G})$ (3.7.1) shows (taking into account (3.4.4)) that the homomorphism of fibres $\psi_x \circ \rho_{\psi(x)} : \mathcal{G}_{\psi(x)} \to (\psi^*(G))_x$ is a functorial isomorphism in $\mathcal{G}$, which allows us to identify the two fibres. Via this identification, $u_x^\sharp$ is precisely the homomorphism defined in (3.5.1) and, in particular, we have $Supp(\psi^*(\mathcal{G})) = \psi^{-1}(Supp(\mathcal{G}))$.

An immediate consequence of this result is that the functor $\psi^*(\mathcal{G})$ is exact on $\mathcal{G}$ in the abelian category of sheaves of abelian groups.

Sheaves of Pseudo-Discrete Spaces

3.8.1 Let X be a topological space whose topology admits a basis $\mathfrak{B}$ consisting of quasi-compact open sets. Let $\mathcal{F}$ be a sheaf of sets on X. If each $\mathcal{F}(U)$ is given the discrete topology then $U \mapsto \mathcal{F}(U)$ is a presheaf of topological spaces. We will see that there is a sheaf of topological spaces $\mathcal{F}'$ associated to $\mathcal{F}$ (3.5.6) such that $\Gamma(U, \mathcal{F}')$ is the dicrete space $\mathcal{F}(U)$ for every open, quasi-compact U. It will suffice to show that the presheaf $U \mapsto \mathcal{F}(U)$ of discrete topological spaces on $\mathfrak{B}$ satisfies condition (F_0) of (3.2.2) and, more generally, if U is a quasi-compact open and if (U_x) is a cover of U by sets in $\mathfrak{B}$ then the coarsest topology $\mathcal{T}$ on $\Gamma(U, \mathcal{F})$ making the maps $\Gamma(U, \mathcal{F}) \to \Gamma(U_\alpha, \mathcal{F})$ continuous is the discrete topology. However, there are a finite number of indices α_i such that $U = \bigcup U_{\alpha_i}$. Let $s \in \Gamma(U, \mathcal{F})$ and let s_i be its image in $\Gamma(U_{\alpha_i}, \mathcal{F})$. The intersection of the inverse images of the sets $\{s_i\}$ is, by definition, a neighborhood of s for $\mathcal{T}$, but since $\mathcal{F}$ is a sheaf of sets and the U_{α_i} cover U, this intersection is $\{s\}$, proving our assertion.

One notes that if U is an open, non quasi-compact subset of X then topological space $\Gamma(U, \mathcal{F}')$ still has the underlying set of $\Gamma(U, \mathcal{F})$, but that the topology is not necessarily discrete; it is the coarsest making the maps $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ continuous for $V \in \mathfrak{B}$ and $V \subseteq U$ ($\Gamma(V, \mathcal{F})$ being discrete).

The preceding considerations apply without modification for sheaves of groups and (not necessarily commutative) rings with their respective sheaves of topological groups and topological rings. In the interest of brevity, we say that the sheaf $\mathcal{F}'$ is the *pseudo−discrete sheaf of spaces* (resp. groups, rings) *associated to* the sheaf of sets (resp. groups, rings) $\mathcal{F}$.

(3.8.2) Let $\mathcal{F}, \mathcal{G}$ be sheaves of sets (resp. groups, rings) on X and $u : \mathcal{F} \to \mathcal{G}$ a homomorphism. Then u is also a continuous homomorphism $\mathcal{F}' \to \mathcal{G}'$, where $\mathcal{F}$ and $\mathcal{G}'$ are the pseudo-discrete sheaves associated to $\mathcal{F}$ and $\mathcal{G}$. This results from (3.2.5).

(3.8.3) Let $\mathcal{F}$ be a sheaf of sets, $\mathcal{H}$ a subsheaf of $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{H}'$ the pseudo-discrete sheaves associated to $\mathcal{F}$ and $\mathcal{H}$ respectively. Then, for all open $U \subseteq X$, $\Gamma(U, \mathcal{H}')$ is closed in $\Gamma(U, \mathcal{F}')$. Indeed, it is the intersection of the inverse images of the $\Gamma(V, \mathcal{H})$ (for

$V \in \mathfrak{B}$, $V \subseteq U$) under the continuous maps $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$, and $\Gamma(V, \mathcal{H})$ is closed in the discrete space $\Gamma(V, \mathcal{F})$.

4. Ringed Spaces

4.1. Ringed Spaces, $\mathcal{A}$-Modules, $\mathcal{A}$-Algebras

(4.1.1) A *ringed* (resp. *topologically ringed*) *space* is a couple $(X, \mathcal{A})$ consisting of a topological space X and a sheaf of (not necessarily commutative) rings (resp. topological rings) $\mathcal{A}$ on X. X is said to be the *underlying topological space* of the ringed space $(X, \mathcal{A})$ and $\mathcal{A}$ the *strcture sheaf*. This last is denoted $\mathcal{O}_X$ and the fibre at a point $x \in X$ is denoted $\mathcal{O}_{X,x}$, or simply $\mathcal{O}_x$ when there is no risk of confusion.

We denote by 1 or e the unit section of $\mathcal{O}_X$ over X (the unit element of $\Gamma(X, \mathcal{O}_X)$).

Since, in this volume, we will need to consider commutative rings we will assume that, unless otherwise specified, that the structure sheaf $\mathcal{A}$ is a sheaf of commutative rings.

The ringed spaces (resp. topologically ringed spaces) with structure sheaf not necessarily commutative form a category, when one defines a morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$ to be a couple $(\psi, \theta) = \Psi$ consisting of a continuous map $\psi : X \to Y$ and a ψ-morphism $\theta : \mathcal{B} \to \mathcal{A}$ (3.5.1) of rings (resp. topological rings). The composition of Ψ and second morphism $\Psi' = (\psi', \theta') : (Y, \mathcal{B}) \to (Z, \mathcal{C})$, denoted $\Psi'' = \Psi' \circ \Psi$, is the morphism (ψ'', θ'') where $\psi'' = \psi' \circ \psi$ and θ'' is the composition of θ and θ' (ie-$\psi'_*(\theta) \circ \theta'$, cf. 3.5.2). For ringed spaces, recall that we then have $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (3.5.5). Thus, if $\theta'^\sharp$ and $\theta^\sharp$ are injective (resp. surjective) homomorphisms, then the same is true of $\theta''^\sharp$ taking into account that $\psi_x \circ \rho_{\psi(x)}$ is an isomorphism for all $x \in X$ (3.7.2). One verifies immediately, thanks to the preceding, that, when ψ is a continuous injective map and $\theta^\sharp$ is a surjective homomorphism of sheaves of rings, the morphism (ψ, θ) is a monomorphism (T, 1.1) in the category of ringed spaces.

By abuse of language, one often replaces ψ by Ψ in notation. For example, we write $\Psi^{-1}(U)$ in place of $\psi^{-1}(U)$ for a subset U of Y when there is no risk of confusion.

(4.1.2) For every subset M of X the couple $(M, \mathcal{A}_{|M})$ is clearly a ringed space, induced on M by the ringed space $(X, \mathcal{A})$ (and still called the restriction of $(X, \mathcal{A})$ to M). If j is the canonical injection $M \to X$ and w the identity map on $\mathcal{A}_{|M}$ then $(j, w^\flat)$ is a monomorphism $(M, \mathcal{A}_{|M}) \to (X, \mathcal{A})$ of ringed spaces called the canonical injection. The composition of a morphism $\Psi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ and this injection is call the *restriction* of Ψ to M.

(4.1.3) We will not revisit the definitions of $\mathcal{A}$-modules or algebraic sheaves on a ringed space $(X, \mathcal{A})$ (G,II,2.2). When $\mathcal{A}$ is a sheaf of (not necessarily commutative) rings, when we say $\mathcal{A}$-module we always mean a left $\mathcal{A}$-module unless otherwise specified. The sub-$\mathcal{A}$-modules of $\mathcal{A}$ will be called *sheaves of ideals* (left, right, or double sided) or $\mathcal{A}$-ideals.

When $\mathcal{A}$ is a sheaf of commutative rings and, in the definition of $\mathcal{A}$-modules, one replaces everywhere the structure of a module with that of an algebra, one obtains the definition of an $\mathcal{A}$-algebra on X. That is to say, a (not necessarily commutative) $\mathcal{A}$-algebra is an $\mathcal{A}$-module, $\mathcal{C}$, endowed with a homomorphism of $\mathcal{A}$-modules $\phi : \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C}$ and a section e over X such that the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow{\phi\otimes 1} & \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \\ \downarrow{\scriptstyle 1\otimes\phi} & & \downarrow{\scriptstyle \phi} \\ \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow[\phi]{} & \mathcal{C} \end{array}$$

commutes and, for every open $U \subseteq X$ and every section $s \in \Gamma(U, \mathcal{C})$, one has $\phi((e_{|U}) \otimes s) = \phi(s \otimes (e_{|U})) = s$. To say that $\mathcal{C}$ is a commutative $\mathcal{A}$-algebra is to say that the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & \mathcal{C} & \end{array}$$

commutes where σ is the canonical symmetry map on the tensor product $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$.

The homomorphisms of $\mathcal{A}$-algebras are defined in the same way as homomorphisms of $\mathcal{A}$-modules were defined in (G,II,2.2), but these do not naturally form an abelian group.

If $\mathcal{M}$ is a sub-$\mathcal{A}$-module of an $\mathcal{A}$-algebra $\mathcal{C}$ then the sub-$\mathcal{A}$-algebra of $\mathcal{C}$ generated by $\mathcal{M}$ is the sum of the images of the homomorphisms $\otimes^n \mathcal{M} \to \mathcal{C}$ (for $n \geq 0$). It is also the sheaf associated to the presheaf $U \mapsto \mathcal{B}(U)$ where $\mathcal{B}(U)$ denotes the subalgebra of $\Gamma(U, \mathcal{C})$ generated by the submodule $\Gamma(U, \mathcal{M})$.

(4.1.4) We say that a sheaf of rings on a topological space X is *reduced* at a point $x \in X$ if the fibre $\mathcal{A}_x$ is a reduced ring (1.1.1). We say that $\mathcal{A}$ is reduced if it is reduced at every point of X. We recall that a ring A is said to be regular if every local ring $A_{\mathfrak{p}}$ (as $\mathfrak{p}$ transverses the set of prime ideals of A) is a regular local ring. We say that a sheaf of rings $\mathcal{A}$ on X is *regular* at a point x (resp. regular) if the fibre $\mathcal{A}_x$ is a regular ring (resp. if $\mathcal{A}$ is regular at every point). Finally, we say that s sheaf of rings $\mathcal{A}$ on X is *normal* at a point x (resp. normal) if the fibre $\mathcal{A}_x$ is integral and integrally closed (resp. if $\mathcal{A}$ is normal at every point). We say that the ringed space $(X, \mathcal{A})$ has one of the preceding properties if the sheaf of rings $\mathcal{A}$ has that property.

A sheaf of graded ring is by definition a sheaf of rings which is the direct sum (G,II,2.7) of a family $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ of sheaves of abelian groups such that $\mathcal{A}_m\mathcal{A}_n \subseteq \mathcal{A}_{m+n}$. A graded $\mathcal{A}$-module $\mathcal{F}$ is an $\mathcal{A}$-module that is the direct sum of a family $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ of abelian groups satisfying $\mathcal{A}_m\mathcal{F}_n \subseteq \mathcal{F}_{m+n}$. It is equivalent to say that $(\mathcal{A}_m)_x(\mathcal{A}_n)_x \subseteq (\mathcal{A}_{m+n})_x$ (resp. $(\mathcal{A}_m)_x(\mathcal{F}_n)_x \subseteq (\mathcal{F}_{m+n})_x$) for every point x.

(4.1.5) Given a (not necessarily commutative) ringed space $(X, \mathcal{A})$, we will not recall the definitions of the bifunctors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ in the categories of left and right $\mathcal{A}$-modules with values in the category of sheaves of abelian groups (or more generally, of $\mathcal{C}$-modules, if $\mathcal{C}$ is the center of $\mathcal{A}$). The fibre $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x$ is canonically identified with $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ and one can define a canonical functorial homomorphism $(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x \to Hom_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ which is not, in general, injective or surjective. The bifunctors considered above are additive and, in particular, commute with finite direct sums. $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is right exact in $\mathcal{F}$ and $\mathcal{G}$ and commutes with inductive limits, and $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{G}$ (resp. $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}$) is canonically identified with $\mathcal{G}$ (resp. $\mathcal{F}$). The functors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ are left exact in $\mathcal{F}$ and $\mathcal{G}$. More precisely, given an exact sequence of the form $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$, the sequence

$$0 \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}') \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}'')$$

is exact, and given an exact sequence of the form $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, the sequence

$$0 \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G})$$

is exact, with the analogous properties for the functor Hom. Moreover, $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{G})$ is canonically identified with $\mathcal{G}$. Finally, for all open $U \subseteq X$, one has

$$\Gamma(U, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) = Hom_{\mathcal{A}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U}).$$

For every left (resp. right) $\mathcal{A}$-module $\mathcal{F}$, one calls the right (resp. left) $\mathcal{A}$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$ the *dual* of $\mathcal{F}$, denoted $\check{\mathcal{F}}$.

Finally, if $\mathcal{A}$ is a sheaf of commutative rings and $\mathcal{F}$ is an $\mathcal{A}$-module then $U \mapsto \wedge^p\Gamma(U, \mathcal{F})$ is a presheaf whose associated sheaf is an $\mathcal{A}$-module which we denote $\wedge^p\mathcal{F}$ and call the p^{th} *exterior power* of $\mathcal{F}$. One easily verifies that the canonical map of presheaves from $U \mapsto \wedge^p\Gamma(U, \mathcal{F})$ to the associated sheaf $\wedge^p\mathcal{F}$ is injective and that, for all $x \in X$, one has $(\wedge^p\mathcal{F})_x = \wedge^p(\mathcal{F}_x)$. It is clear that $\wedge^p\mathcal{F}$ is a covariant functor in $\mathcal{F}$.

(4.1.6) Suppose that $\mathcal{A}$ is a sheaf of (not necessarily commutative) rings, $\mathcal{J}$ is a sheaf of left ideals of $\mathcal{A}$, and $\mathcal{F}$ is a left $\mathcal{A}$-module. One denotes by $\mathcal{J}\mathcal{F}$ the sub-$\mathcal{A}$-module of $\mathcal{F}$ that is the image of $\mathcal{J} \otimes_{\mathbb{Z}} \mathcal{F}$ (where $\mathbb{Z}$ is the sheaf associated to the constant presheaf $U \mapsto \mathbb{Z}$) under the canonical map $\mathcal{J} \otimes_{\mathbb{Z}} \mathcal{F} \to \mathcal{F}$. It is clear that, for all $x \in X$, we have $(\mathcal{J}\mathcal{F})_x = \mathcal{J}_x\mathcal{F}_x$. When $\mathcal{A}$ is commutative $\mathcal{J}\mathcal{F}$ is also the canonical image of $\mathcal{J} \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$. It is immediate that $\mathcal{J}\mathcal{F}$ is also the $\mathcal{A}$-module associated to the presheaf $U \mapsto \Gamma(U, \mathcal{J})\Gamma(U, \mathcal{F})$. If $\mathcal{J}_1$, $\mathcal{J}_2$ are two sheaves of left ideals of $\mathcal{A}$, one has $\mathcal{J}_1(\mathcal{J}_2\mathcal{F}) = (\mathcal{J}_1\mathcal{J}_2)\mathcal{F}$.

(4.1.7) Let $(X_\lambda, \mathcal{A}_\lambda)_{\lambda\in L}$ be a family of ringed spaces. For every couple (λ, μ) suppose we have an open subset $V_{\lambda\mu}$ of X and an isomorphism of ringed spaces $\varphi_{\lambda\mu} : (V_{\mu\lambda}, \mathcal{A}_{\mu|V_{\mu\lambda}}) \to (V_{\lambda\mu}, \mathcal{A}_{\lambda|V_{\lambda\mu}})$ with $V_{\lambda\lambda} = X_\lambda$ and $\varphi_{\lambda\lambda}$ the identity. Suppose moreover that, for every triplet (λ, μ, ν), if one denotes by $\varphi'_{\mu\lambda}$ the restriction of $\varphi_{\mu\lambda}$ to $\varphi_{\lambda\mu} \cap \varphi_{\lambda\nu}$ then $\varphi'_{\mu\lambda}$ is an isomorphism of $(V_{\lambda\mu} \cap V_{\lambda\nu}, \mathcal{A}_{\lambda|V_{\lambda\mu}\cap V_{\lambda\nu}})$ and $(V_{\mu\nu} \cap V_{\mu\lambda}, \mathcal{A}_{\mu|V_{\mu\nu}\cap V_{\mu\lambda}})$ and one has $\varphi'_{\lambda\nu} = \varphi'_{\lambda\mu} \circ \varphi'_{\mu\nu}$ (the gluing condition for the $\varphi_{\lambda\mu}$). One first of all considers the topological space obtained by gluing the X_λ along the $V_{\lambda\mu}$ (using the $\varphi_{\lambda\mu}$). If one identifies X_λ with its corresponding open subset X'_λ of X, the assumptions imply that the three sets $V_{\lambda\mu} \cap V_{\lambda\nu}$, $V_{\mu\nu} \cap V_{\mu\lambda}$, and $V_{\nu\lambda} \cap V_{\nu\mu}$ are identified with $X'_\lambda \cap X'_\mu \cap X'_\nu$. One can then transfer the ringed space structure of X_λ to X'_λ and, if $\mathcal{A}'_\lambda$ is the sheaf of rings transferred from $\mathcal{A}_\lambda$ then $\mathcal{A}'_\lambda$ satisfies the gluing conditions (3.3.1) and defines a sheaf of rings $\mathcal{A}$ on X. We call $(X, \mathcal{A})$ the ringed space obtained by *gluing the* $(X_\lambda, \mathcal{A}_\lambda)$ *along the* $V_{\lambda\mu}$ *via the* $\varphi_{\lambda\mu}$.

4.2. Direct Image of an $\mathcal{A}$-module

(4.2.1) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be ringed spaces and $\Psi = (\psi, \theta)$ a morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$. $\psi_*(\mathcal{A})$ is then a sheaf of rings on Y and θ is a homomorphism $\mathcal{B} \to \psi_*(\mathcal{A})$ of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{A}$-module. Its direct image $\psi_*(\mathcal{F})$ is a sheaf of abelian groups on Y. Moreover, for every open $U \subseteq Y$,

$$\Gamma(U, \psi_*(\mathcal{F})) = \Gamma(\psi^{-1}(U), \mathcal{F})$$

is endowed with a module structure via $\Gamma(U, \psi_*(\mathcal{A})) = \Gamma(\psi^{-1}(U), \mathcal{A})$. The bilinear maps defining these structures are compatible with the restriction maps, defining on $\psi_*(\mathcal{F})$ the structure of a $\psi_*(\mathcal{A})$-module. The homomorphism $\theta : \mathcal{B} \to \psi_*(\mathcal{A})$ then allows us to define the structure of a $\mathcal{B}$-module on $\psi_*(\mathcal{F})$. We will call this $\mathcal{B}$-module the *direct image of* $\mathcal{F}$ by the morphism Ψ and denote it $\Psi_*(\mathcal{F})$. If $\mathcal{F}_1$, $\mathcal{F}_2$ are $\mathcal{A}$-modules on X and $u : \mathcal{F}_1 \to \mathcal{F}_2$ is an $\mathcal{A}$-module homomorphism, it is immediate (considering sections over opens of Y) that $\psi_*(u)$ is a $\psi_*(\mathcal{A})$-homomorphism $\psi_*(\mathcal{F}_1) \to \psi_*(\mathcal{F}_2)$ and, a fortiori, a $\mathcal{B}$-homomorphism $\Psi_*(\mathcal{F}_1) \to \Psi_*(\mathcal{F}_2)$ which we denote $\Psi_*(u)$. Thus, Ψ_* is a covariant functor from the category of $\mathcal{A}$-modules to the category of $\mathcal{B}$-modules. Moreover, it is immediate that this functor is left exact (G,II,2.12).

We define the structure of a $\mathcal{B}$-algebra on $\psi_*(\mathcal{A})$ via its structure as a $\mathcal{B}$-module and its structure as a sheaf of rings. We denote this $\mathcal{B}$-algebra $\Psi_*(\mathcal{A})$.

(4.2.2) Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-modules. For every open U of Y, there is a canonical map

$$\Gamma(\psi^{-1}(U), \mathcal{M}) \times \Gamma(\psi^{-1}(U), \mathcal{N}) \to \Gamma(\psi^{-1}(U), \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

which is bilinear over the ring $\Gamma(\psi^{-1}(U), \mathcal{A}) = \Gamma(U, \psi_*(\mathcal{A}))$ and, a fortiori, over $\Gamma(U, \mathcal{B})$. It thus defines a homomorphism

$$\Gamma(U, \Psi_*(\mathcal{M})) \otimes_{\Gamma(U,\mathcal{B})} \Gamma(U, \Psi_*(\mathcal{N})) \to \Gamma(U, \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}))$$

and, as one verifies immediately that these homomorphisms are compatible with the restriction maps, they give a canonical functorial homomorphism of $\mathcal{B}$-modules

$$(4.2.2.1)\ \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N}) \to \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

which is not, in general, injective or surjective. If $\mathcal{P}$ is a third $\mathcal{A}$-module, one verifies immediately that the diagram

$$(4.2.2.2)\quad \begin{array}{ccc} \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N}) \otimes_{\mathcal{B}} \Psi * (\mathcal{P}) & \longrightarrow & \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{P}) \\ \downarrow & & \downarrow \\ \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N} \otimes_{\mathcal{A}} \mathcal{P}) & \longrightarrow & \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{P}) \end{array}$$

commutes.

(4.2.3) Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-modules. For every open $U \subseteq Y$, one has by definition $\Gamma(\psi^{-1}(U), \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) = Hom_{\mathcal{A}_{|V}}(\mathcal{M}_{|V}, \mathcal{N}_{|V})$ and one can claim $V = \psi^{-1}(U)$. The map $u \mapsto \Psi_*(u)$ is a homomorphism

$$Hom_{\mathcal{A}_{|V}}(\mathcal{M}_{|V}, \mathcal{N}_{|V}) \to Hom_{\mathcal{B}_{|U}}(\Psi_*(\mathcal{M})_{|U}, \Psi_*(\mathcal{N})_{|U})$$

of $\Gamma(U, \mathcal{B})$-modules. These homomorphisms, being compatible with the restriction maps, define a canonical functorial homomorphism of $\mathcal{B}$-modules

$$(4.2.3.1)\ \Psi_*(\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) \to \mathcal{H}om_{\mathcal{B}}(\Psi_*(\mathcal{M}), \Psi_*(\mathcal{N})).$$

(4.2.4) If $\mathcal{C}$ is an $\mathcal{A}$-algebra then the composition

$$\Psi_*(\mathcal{C}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{C}) \to \Psi_*(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}) \to \Psi_*(\mathcal{C})$$

gives $\Psi_*(\mathcal{C})$ the structure of a $\mathcal{B}$-algebra as a result of (4.2.2.2). Similarly, one sees that, if $\mathcal{M}$ is a $\mathcal{C}$-module, then $\Psi_*(\mathcal{M})$ is (canonically) a $\Psi_*(\mathcal{C})$-module.

(4.2.5) Consider, in particular, the case where X is a closed subspace of Y and ψ is the canonical injection $j : X \to Y$. If $\mathcal{B}' = \mathcal{B}_{|X} = j^*(\mathcal{B})$ is the restriction of the sheaf of rings $\mathcal{B}$ to X then an $\mathcal{A}$-module $\mathcal{M}$ can be regarded as a $\mathcal{B}'$-module via the homomorphism $\theta^\sharp : \mathcal{B}' \to \mathcal{A}$. $\Psi_*(\mathcal{M})$ is then a $\mathcal{B}$-module which induces $\mathcal{M}$ on X and 0 elsewhere. If $\mathcal{N}$ is a second $\mathcal{A}$-module then $\Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{M})$ is identified with $\Psi_*(\mathcal{M} \otimes_{\mathcal{B}'} \mathcal{N})$ and $\mathcal{H}om_{\mathcal{B}}(\Psi_*(\mathcal{M}), \Psi_*(\mathcal{N}))$ with $\Psi_*(\mathcal{H}om_{\mathcal{B}'}(\mathcal{M}, \mathcal{N}))$.

(4.2.6) Let $(Z, \mathcal{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathcal{B}) \to (Z, \mathcal{C})$. If Ψ'' is the composition $\Psi' \circ \Psi$ then clearly $\Psi''_* = \Psi'_* \circ \Psi_*$.

4.3. The Inverse Image of a $\mathcal{B}$-Module

(4.3.1) Under the hypotheses of (4.2.1), let $\mathcal{G}$ be a $\mathcal{B}$-module and $\psi^*(\mathcal{G})$ its inverse image (3.7.1), which is a sheaf of abelian groups on X. The definition of the sections of $\psi^*(\mathcal{G})$ and $\psi^*(\mathcal{B})$ (3.7.1) shows that $\psi^*(\mathcal{G})$ is canonically endowed with the structure of a $\psi^*(\mathcal{B})$-module. In addition, the homomorphism $\theta^\sharp : \psi^*(\mathcal{B}) \to \mathcal{A}$ provides $\mathcal{A}$ with the structure of a $\psi^*(\mathcal{B})$-module, which is denoted $\mathcal{A}_{[\theta]}$ when necessary to avoid confusion. The tensor product $\psi^*(\mathcal{G}) \otimes_{\psi_*(\mathcal{B})} \mathcal{A}_{[\theta]}$ is then endowed with the structure of an $\mathcal{A}$-module. We will say that this module is the *inverse image* of $\mathcal{G}$ by the morphism Ψ, denoted $\Psi^*(\mathcal{G})$.