ULTRAPRODUCTS IN COMMUTATIVE ALGEBRA

MATH 83600

Textbook. *The use of ultraproducts in commutative algebra* [7]. I will complement this with some additional course notes, to be posted on my website, on the topics (2)–(4) below from commutative algebra.

Prerequisites. I will present all concepts in sufficient detail, so that not too much background is needed. However, a first course in algebra is required. In particular, you should be familiar with basic ring and module theory. For instance, a good introduction is [1], and I will assume knowledge of the material in Chapters 1-3, 6-8 and 10, and preferably also Chapter 11.

More advanced topics in commutative algebra—such as dimension theory, singularities, flatness, completion—will be discussed in detail and no prior knowledge is assumed (apart from the notes, the main references will be [2, 4]). For certain topics some knowledge of homological algebra and/or category theory might be useful, but is not really necessary. Although we use a notion from model-theory, to wit, ultraproducts, we will approach everything in algebraic terms and so no prior knowledge of model-theory is really necessary (but I would recommend [5, Chapters 1-4]), and neither is any prior exposure to algebraic geometry. It will be one of our main goals to discuss in depth the beautiful theory of tight-closure; references are [3, 6] for respectively the positive characteristic and zero characteristic case.

Course outline. The following outline is probably too ambitious, and we will have to skip some topics.

(1) Ultraproducts and the Lefschetz Principle: ultrafilters on $\mathbb{N}$; ultraproducts of rings; Łos' Theorem (equational version); Lefschetz Principle for algebraically closed fields.
(2) Algebraic Geometry versus Commutative Algebra: affine varieties; coordinate ring of an affine variety; local ring of a point; spectrum of a ring.
(3) Dimension theory: Krull dimension; Hilbert functions.
(4) Singularities: regular local rings; regular sequences; Cohen-Macaulay local rings.
(5) Flatness I: tensor products; exact sequences; flatness.
(6) Flatness II: the Tor groups; flatness criteria.
(7) Schmidt-van den Dries Theorem; uniform bounds.
(8) Tight closure theory I: Frobenius in characteristic p; integral closure; tight closure.
(9) Tight closure theory II: Kunzs theorem; F-regularity; colon capturing; F-rationality.
(10) Applications of tight closure: Hochster-Roberts Theorem; Briançon-Skoda Theorem.

(11) Tight closure theory in characteristic zero I: ultra-Frobenius; non-standard hull of an algebra; non-standard tight closure.
(12) Lefschetz rings: completion of a local ring; Artin Approximation; embedding theorem.
(13) Tight closure theory in characteristic zero II: tight closure for arbitrary Noetherian local rings; big Cohen-Macaulay algebras.
(14) Cataproducts: infinitesimals; cataproducts; flatness of catapowers; uniform bounds.

References

1. M. Atiyah and G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass., 1969.
2. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
3. C. Huneke, *Tight closure and its applications*, CBMS Regional Conf. Ser. in Math, vol. 88, Amer. Math. Soc., 1996.
4. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
5. P. Rothmaler, *Introduction to model theory*, Algebra, Logic and Applications, vol. 15, Gordon and Breach Science Publishers, Amsterdam, 2000.
6. Hans Schoutens, *Characteristic p methods in characteristic zero via ultraproducts*, Recent Developments in Commutative Algebra (B. Olberding M. Fontana, S. Kabbaj and I. Swanson, eds.), Springer-Verlag, 2010.
7. ______, *The use of ultraproducts in commutative algebra*, Lecture Notes in Mathematics, vol. 1999, Springer-Verlag, 2010.