MATH 87800: Differential Algebra [18718]
Thursday, 9:30am - 11:30am, 3 cr.
Prof. Alexey Ovchinnikov

COURSE DESCRIPTION

There will be two main topics in the course:

1) Basics of differential ideal theory, differential elimination, and differentially closed fields;

2) Basics of differential Galois theory and linear algebraic groups.

Differential ideal theory can tell whether a system of algebraic PDEs (when the equations are polynomial in the derivatives) is consistent or, more generally, test if a given PDE is a consequence of the system. This theory can also be applied to simplification of these systems by, for instance, elimination of variables. This is an analogue of Gaussian elimination for systems of linear equations and of Groebner bases for polynomial equation. We will discuss basic theorems including the Ritt basis theorem (analogue of the Hilbert basis theorem), differentially closed fields, differential Nullstellensatz (including its effective version and the use of ultraproducts in it), and decomposition of radical differential ideals (these ideals correspond to solutions of systems of algebraic PDEs) into prime components.

Decomposing radical differential ideals into prime components is a very important algorithm that exhibits essential properties of differential equations. Over 50 years ago, Ritt posed his famous
problem, which is an algebraic treatment of the following analytic phenomenon occurring with differential equations. A calculation shows that the solution set to the differential equation

y'^2-4y = 0

consists of a family of parabolae and the zero special solution.

The solution set to the equation

y'^2-4y^3 = 0

consists of hyperbolae and the zero solution as well. The major difference between these two solution sets is that, in the first case, the zero solution is not a limit of the family while, in the second case, it is. In the algebraic language, the first equation does not generate a prime differential ideal, while the second one does. The Ritt problem is a generalization of this phenomenon to the case of several non-linear algebraic PDEs and is equivalent to finding an irredundant prime decomposition of a radical differential ideal. This problem (irredundancy) still remains open, however just recently new techniques have been created to come closer to solving it.

The Galois theory of differential equations was begun in the 19th century by Picard and Vessiot and further developed in the 20th century by Kolchin and others. This theory allows one to associate to any linear differential equation a group of matrices. This group of matrices turns out to be the collection of matrices satisfying a certain set of polynomial equations in the entries (it is a linear algebraic group). Properties of solutions of the equation (e.g., solvability in terms of special functions) are reflected in properties of this group. We will discuss basic theorems including the Galois correspondence between intermediate subfields and subgroups of the Galois group; correspondence between solvable groups and differential equations solvable in terms of integrals, exponentiation, and algebraic functions (with Liouvillian solutions); and if time permits, algorithms computing Galois groups.

Existing algorithms that compute differential Galois groups can calculate solutions at the same time if the equation is solvable in the above sense. In this theory, all first order systems are solvable. Many of second order systems are solvable as well. The simplest example of a non-solvable ODE over C(x) is the Airy equation:

$y''=xy$;

its differential Galois group is not solvable, it is SL(2).

Currently, there are plenty of algorithms computing Galois groups, and they are now used, for example, for studying such questions as integrability of Hamiltonian systems.

Main references:

1) I. Kaplansky, "An introduction to differential algebra",
Hermann, Paris, 1957.

2) M. van der Put and M. Singer, "Galois theory of linear
differential equations", Springer-Verlag, Berlin, 2003.
http://www4.ncsu.edu/~singer/papers/dbook2.ps