

# Debt as Safe Asset\*

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## Abstract

The price of a safe asset reflects not only the expected discounted future cash flows but also future service flows, since retrading allows partial insurance of idiosyncratic risk in an incomplete markets setting. This lowers the issuers' interest burden and allows the government to run a permanent (primary) deficit without ever paying back its debt, like a Ponzi scheme, while individual citizens' transversality conditions hold. As idiosyncratic risk rises during recessions, so does the value of the service flows bestowing the safe asset with a negative  $\beta$ . This resolves government debt valuation puzzles. Nevertheless, the government faces a "Debt Laffer Curve" and the paper has important implications for fiscal debt sustainability and the FTPL.

**Keywords:** Safe Asset, Government Debt, Debt Laffer Curve, FTPL, Fiscal Capacity, I Theory of Money

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# 1 Introduction

How much government debt can the market absorb? At what interest rate? Is there a limit, a “Debt Laffer Curve”? When can governments run a permanent (primary) deficit without ever paying back its debt, like a Ponzi scheme, and nevertheless individual citizens’ transversality conditions hold? What is a safe asset? What are its features? Why is government debt a safe asset? When does one lose the safe asset status? Why is there debt valuation puzzle for governments for advanced countries like the US and Japan? How do we have to modify representative agent asset pricing and the Fiscal Theory of the Price Level (FTPL) equation?

This paper attempts to address these questions within a setting in which citizens face uninsurable idiosyncratic risks and hence save for precautionary reasons. Each citizen lives forever and adjusts his portfolio consisting of physical capital and the government bond. Idiosyncratic shocks that cannot be diversified away (as well as aggregate shocks) make capital risky. This makes government bonds attractive since they can be sold after an adverse shock. From an individual citizen’s perspective it is this ability to retrade, which makes the government bond a desirable hedging instrument. His planned dynamic trading strategy generates a payoff stream that is a good hedge. This is the first of the two key characteristics of a safe asset, the *Good Friend Analogy*.<sup>1</sup> A safe asset is like a good friend, it is around; that is, it is (i) valuable and (ii) liquid when one needs it.<sup>2</sup> The government bond is a safe asset and is desirable even if it does not yield any interest or dividend.

In other words, the classic asset pricing equation consists not only of the appropriately discounted cash flow stream but has to be complemented with a discounted stream of service flows. The retrading allows citizens in the economy to partially insure each other and overcome the incomplete markets friction. Hence, the real value of government debt, i.e. the nominal value  $\mathcal{B}$  divided by the price level  $\mathcal{P}$  is

$$\mathcal{B}/\mathcal{P} = \mathbb{E}[PV[\textit{primary surpluses}]] + \mathbb{E}[PV[\textit{service flows}]].$$

In the same spirit, we propose to include the additional second term also in the FTPL

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<sup>1</sup>The two key characteristics of a safe asset were first proposed in Brunnermeier and Haddad (2012).

<sup>2</sup>Hence, it makes sense for central banks to act as market maker of last resort to ensure that bid-ask spreads remain low. Viewed this way John Law’s big achievement was to create a safe asset status for English and French government debt early in the 18<sup>th</sup> century.

equation.

Importantly, from an individual perspective the transversality condition holds since his discount factor also reflects the idiosyncratic risk she cannot hedge away. On the other hand, from the aggregate perspective, the issuer's perspective, the safe asset component of the asset pricing equation is a bubble.

When adding aggregate shocks the full feature of safe assets emerges. We consider economies when entering a recession, aggregate output declines and at the same time idiosyncratic risk rises. Let us consider both components of the asset pricing equation, which also underlines the FTPL. The first term reflects the mainstream view, prevalent in the representative agent asset pricing. A drop in output reduces payoffs and increases the marginal utility, leading to the traditional positive  $\beta$  in the asset pricing equation. The second term, the safe asset term, captures the discounted stream of service flows, which in our setting yields partial insurance benefit. This term behaves very differently. As idiosyncratic risk rises in recessions, citizens prefer to shift their portfolio away from capital towards the government bond, resulting in a force that pushes up the real value of government debts. That is, the second term due to the discounted stream of service flows has a negative  $\beta$ .<sup>3</sup> In a sense, the Jiang et al. (2019)'s "debt valuation puzzle" for the US can be seen as an empirical vindication of the importance of the second term in our analysis. Even more pronounced the primary surplus in Japan was negative for 50 out of the last 60 years, also suggesting a large second term overpowering the first term.

The second characteristic feature of safe assets is the *Safe Asset Tautology*. A safe asset is safe when it is perceived to be safe so that in times of crisis investors flock to it. In other words, the safe asset status is highly endogenous and part of a multiple equilibrium structure. From an aggregate perspective a safe asset is a "bubble" and bubbles can pop. As a consequence, an asset can lose its safe asset status. Government debt is special as long as the government has sufficient fiscal space to fend off a possible jump to a non-safe asset equilibrium. Note that the ability alone to permanently raise taxes to back the debt is sufficient to prevent such a jump. This ability should be an important element in any debt sustainability analysis. Private companies do not have taxing

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<sup>3</sup>The second term can also be due to binding collateral constraints, (narrow) money as medium of exchange benefits or any other form of convenience yield as in Krishnamurthy and Vissing-Jorgensen (2012). However, it is not obvious whether the last two forms of convenience yield would generate a negative  $\beta$ .

power and hence can not fully replicate the off-equilibrium backing. An explicit and formal characterization of the fragility of the safe asset status goes beyond the scope of this paper. In Brunnermeier et al. (2021) we discuss this in the context of an international framework for emerging market economies. Emerging market government bonds' safe asset status competes with advanced economies safe assets and hence are deeply affected by spillovers from US monetary policy.

As long as the safe asset status can be maintained, the government can issue debt at favorable interest rates. Citizens are willing to receive a low interest rate since they enjoy the service flow. Indeed, the government can even run a sustainable Ponzi scheme: Pay off the maturing bonds with newly issued debt and issuing more for additional expenditures. In other words, the government can "mine the bubble". As the growth rate of the supply of bonds increases, the citizens' cash-flow return of holding the government bonds declines. "Printing" bonds at a faster rate acts like a tax on bond holdings and consequently lowers the "tax base", the value of the bonds. A "*Debt Laffer Curve*" arises. When tax exceeds a certain level overall tax revenue from bubble mining declines. As the government issues bonds at a rate so high that the price of bonds collapses, the tax revenue vanishes as well.

Overall, a safe asset perspective sheds a different light on the valuation of government debt and stresses that any debt sustainability analysis (DSA) should include the fragility of the safe asset status.

Our model has also interesting stock market asset pricing implications due to "flight-to safety" phenomena. During recessions idiosyncratic risk is assumed to rise. While for outside equity idiosyncratic risk can be diversified away, each household who manages her firm is exposed to her idiosyncratic risk via her inside equity holding. Hence, each citizen demands a higher risk premium during recession which depresses her demand for the (outside) equity stock index in favor of demand for the safe asset.

**Literature.** This paper touches upon many strands of classic and recent economic literature. We follow the safe asset definition outlined in Brunnermeier and Haddad (2012). Dang et al. (2015) emphasize the information insensitivity of safe assets. See also Gorton (2010), Stein (2012), Moreira and Savov (2014), Gorton and Ordóñez (2013, 2014), Dang, Gorton and Holmström (2015) and Greenwood, Hanson and Stein (2015). In He et al. (2019) model safe asset tautology within a generalized global games setting. Caballero et al. (2017) stress the importance of safe asset shortage. Brunnermeier et al.

(2016, 2017) proposes a safe asset via securitization and argues that the main problem is the asymmetric supply of safe assets leading to eruptive cross-border capital flows.

This paper resolves the “Public Debt Valuation Puzzle” proposed in Jiang et al. (2019), which argues that the value of government debt should be significantly lower not least because since primary surpluses, the total payments to all bond holders, are countercyclical. In our setting the price of debt is procyclical since the bubble-term rises in bad times, resulting in a negative  $\beta$  asset. Second, it also resolves the “Government Debt Risk Premium Puzzle” (Jiang et al., 2020), the puzzle that the government debt appears to insure simultaneously bond holders and taxpayers whereas in standard models it can insure only one of the two groups. Our analysis shows that the bubble term can make the bond a negative  $\beta$ -asset, a good hedge for bond holders, while primary surpluses are countercyclical at the same time, thus providing insurance for taxpayers. Surprisingly, traded equity in our model exhibits excess volatility and predictability.

The value of government debt is inherently linked with fiscal debt sustainability. In deterministic models debt is sustainable and a Ponzi scheme is feasible if the risk free interest rate  $r$  is lower than the economic growth rate  $g$ . Bohn (1995) questions the simple  $r$  vs.  $g$  comparison for economies with aggregate risk. Overlapping Generations (OLG) include Samuelson (1958), Diamond (1965) with capital, Tirole (1985) with a bubble and most recently by Blanchard (2019). Models in which the risk-free rate is depressed due to uninsurable idiosyncratic risk include Bewley (1980), Aiyagari and McGrattan (1998), which calibrates the optimal debt level in a Aiyagari-type model without aggregate risk. Angeletos (2007) studies idiosyncratic investment risks. In Brunnermeier and Sannikov (2016b,a) include a ‘bubbly’ safe asset in the form of government debt or money and allow for aggregate risk. Bassetto and Cui (2018) and Brunnermeier et al. (2020) expand the FTPL equation with a bubble. Reis (2020) also studies fiscal debt capacity for economies in which  $r < g$ , but the marginal return of capital  $m > g$ . Reis (2020) derives the level of bubble mining at which debt value is zero, i.e. when Debt Laffer Curve crosses zero. In Di Tella (2020) money yields additional utility. In Kiyotaki and Moore (2008) citizens self-insure against investment opportunity shocks. There is an extensive literature on rational bubbles. Survey papers include Miao (2014) and Martin and Ventura (2018).

## 2 Model

### 2.1 Model Setup

There is a continuum of households indexed by  $i \in [0, 1]$ . All households have identical logarithmic preferences

$$\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \log c_t^i dt \right]$$

with discount rate  $\rho$ .<sup>4</sup>

Each agent operates one firm that produces an output flow  $a_t k_t^i dt$ , where  $k_t^i$  is the capital input chosen by the firm and  $a_t$  is an exogenous productivity process that is common for all agents. Capital of firm  $i$  evolves according to

$$\frac{dk_t^i}{k_t^i} = \left( \Phi \left( i_t^i \right) - \delta \right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_t^{k,i},$$

where  $d\Delta_t^{k,i}$  represents firm  $i$ 's market transactions in physical capital,  $i_t^i k_t^i dt$  are the firm's physical investment expenditures (in output goods),  $\Phi$  is a concave function that captures adjustment costs in capital accumulation,  $\delta$  is the depreciation rate, and  $\tilde{Z}^i$  is an agent-specific Brownian motion that is i.i.d. across agents  $i$ .  $\tilde{Z}^i$  introduces firm-specific idiosyncratic risk.  $\tilde{\sigma}_t$  is an exogenous process that governs the magnitude of idiosyncratic risk faced by agents. To obtain simple closed-form expressions, we choose the functional form  $\Phi(\iota) = \frac{1}{\phi} \log(1 + \phi\iota)$  with adjustment cost parameter  $\phi \geq 0$  for the investment technology.

Each agent  $i$  can sell off some of the risky cash flows generated by capital  $k^i$  to capital markets as outside equity. Outside equity claims on  $i$ 's capital have the same aggregate and idiosyncratic risk as capital itself, but may pay a lower expected return, reflecting an insider premium that  $i$  earns for managing the capital stock. Agents can hold a diversified equity portfolio and thereby eliminate idiosyncratic risk.

The key friction in the model is that agents are unable to share idiosyncratic risk perfectly. Specifically, we assume that agents face a skin-in-the-game constraint and

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<sup>4</sup>In Appendix A.2, we present a generalization with Duffie and Epstein (1992) preferences (continuous time Epstein and Zin (1989) preferences).

must retain at least a fraction  $\bar{\chi} \in (0, 1]$  of their capital in undiversified form, i.e. they can sell off at most a fraction  $1 - \bar{\chi}$  of the cash flows generated by capital  $k^i$  as outside equity. As a consequence, agents have to bear the residual idiosyncratic risk  $\bar{\chi}\tilde{\sigma}_t d\tilde{Z}^i$  inherent in their physical capital holdings.

Besides this limit on idiosyncratic risk sharing, there are no further financial frictions. Agents are allowed to trade physical capital and any type of claim contingent on aggregate risk.

In addition to households, there is a government that funds government spending, imposes taxes on firms, and issues nominal government bonds. The government has an exogenous need for real spending  $g_t K_t dt$ , where  $K_t$  is the aggregate capital stock and  $g_t$  is an exogenous process. The government imposes a proportional output tax (subsidy, if negative)  $\tau_t$  on firms. Outstanding nominal government debt has a face value of  $B_t$  and pays nominal interest  $i_t$ .  $B_t$  follows a continuous process  $dB_t = \mu_t^B B_t dt$ , where the growth rate  $\mu_t^B$  is a policy choice of the government. In short, the government chooses the policy instruments  $\tau_t, i_t, \mu_t^B$  contingent on histories of prices taking  $g_t$  as given and subject to the nominal budget constraint

$$i_t B_t + \mathcal{P}_t g_t K_t = \mu_t^B B_t + \mathcal{P}_t \tau_t a_t K_t, \quad (1)$$

where  $\mathcal{P}_t$  denotes the price level.

We assume that the exogenous processes  $a_t, \tilde{\sigma}_t, g_t$  follow a joint Markov diffusion process that is driven by some Brownian motion  $Z_t$ , which captures aggregate risk and is independent of all the idiosyncratic Brownian motions  $\tilde{Z}_t^i$ .

The model is closed by the aggregate resource constraint

$$C_t + g_t K_t + \iota_t K_t = a_t K_t, \quad (2)$$

where  $C_t := \int c_t^i di$  is aggregate consumption and  $\iota_t = \int \iota_t^i k_t^i / K_t di$  is the average investment rate.

## 2.2 Model Solution

**Price Processes and Returns.** Let  $q_t^K$  be the market price of a single unit of physical capital. Then,  $q_t^K K_t$  is private capital wealth. Let further  $q_t^B := \frac{B_t / \mathcal{P}_t}{K_t}$  be the ratio of the

real value of government debt to total capital in the economy.<sup>5</sup> Then, the real value of the total stock of government bonds is  $q_t^B K_t$  and the real value of a single government bond is  $\frac{q_t^B K_t}{B_t}$ . It is convenient to define the share of total wealth in the economy that is due to bond wealth,

$$\vartheta_t := \frac{q_t^B K_t}{(q_t^B + q_t^K) K_t}.$$

We postulate that  $q_t^B$  and  $q_t^K$  have a generic Ito evolution

$$dq_t^B = \mu_t^{q,B} q_t^B dt + \sigma_t^{q,B} q_t^B dZ_t, \quad dq_t^K = \mu_t^{q,K} q_t^K dt + \sigma_t^{q,K} q_t^K dZ_t.$$

Whenever  $q_t^B, q_t^K \neq 0$ , the unknown (geometric) drifts  $\mu_t^{q,B}, \mu_t^{q,K}$  and volatilities  $\sigma_t^{q,B}, \sigma_t^{q,K}$  are uniquely determined by the local behavior of  $q_t^B$  and  $q_t^K$ , respectively. In the following, we also use the notation  $\mu_t^\vartheta$  and  $\sigma_t^\vartheta$  for the (geometric) drift and volatility of  $\vartheta_t$ .<sup>6</sup>

Households can trade two assets in positive net supply (if  $q_t^B \neq 0$ ), bonds and capital. Assume that in equilibrium  $\iota_t = \iota_t^i$  for all  $i$  (to be verified below) such that aggregate capital grows locally deterministically at rate  $\Phi(\iota_t) - \delta$ . Then, the return on bonds is

$$\begin{aligned} dr_t^B &= i_t dt + \frac{d(q_t^B K_t / B_t)}{q_t^B K_t / B_t} = \frac{d(q_t^B K_t)}{q_t^B K_t} - \overbrace{(\mu_t^B - i_t)}^{=: \check{\mu}_t^B} dt \\ &= \left( \Phi(\iota_t) - \delta + \mu_t^{q,B} - \check{\mu}_t^B \right) dt + \sigma_t^{q,B} dZ_t. \end{aligned} \quad (3)$$

The return on agent  $i$ 's capital is

$$\begin{aligned} dr_t^{K,i}(\iota_t^i) &= \frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \frac{d(q_t^K k_t^i)}{q_t^K k_t^i} \\ &= \left( \frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \Phi(\iota_t^i) - \delta + \mu_t^{q,K} \right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i. \end{aligned}$$

<sup>5</sup>It is more convenient to work with this normalized version of the inverse price level  $1/\mathcal{P}_t$ , because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas  $q_t^B$  does not.

<sup>6</sup>This means,  $d\vartheta_t = \mu_t^\vartheta \vartheta_t dt + \sigma_t^\vartheta \vartheta_t dZ_t$ .



Using the government budget constraint (1) to substitute out  $\tau_t a$  yields

$$dr_t^{K,i} \left( l_t^i \right) = \left( \frac{a_t - \mathfrak{g}_t - l_t^i}{q_t^K} + \frac{q_t^B}{q_t^K} \tilde{\mu}_t^B + \Phi \left( l_t^i \right) - \delta + \mu_t^{q,K} \right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^i.$$

Outside equity claims issued by household  $i$  have the same risk characteristics as the capital return  $dr_t^{K,i}$  but may have a different expected return. The return on outside equity issued by agent  $i$  is therefore

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \sigma_t^{q,K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^i,$$

where the expected return component  $\mathbb{E}_t[dr_t^{E,i}]$  is determined in equilibrium. In equilibrium, all agents optimally hold a perfectly diversified equity portfolio. The return on that portfolio is

$$d\bar{r}_t^E = \int_0^1 r_t^{E,i} di = \mathbb{E}_t[d\bar{r}_t^E] + \sigma_t^{q,K} dZ_t.$$

Because all individual varieties of outside equity  $dr_t^{E,i}$  generate the same aggregate risk contribution to the overall equity portfolio, it will be the case in equilibrium that  $\mathbb{E}_t[dr_t^{E,i}] = \mathbb{E}_t[d\bar{r}_t^E]$  for all  $i$ .

**Household Problem and Equilibrium.** We formulate the household problem as a standard consumption-portfolio-choice problem that does not make explicit reference to the capital trading process  $d\Delta_t^{k,i}$  as a choice variable. For this purpose, denote by  $n_t^i$  the net worth of household  $i$  and let  $\theta_t^{k,i}$ ,  $\theta_t^{E,i}$ ,  $\theta_t^{\bar{E},i}$  be the fraction of net worth invested into capital, own outside equity, and the diversified portfolio of equity, respectively.<sup>7</sup> Then net worth evolves according to

$$\begin{aligned} \frac{dn_t^i}{n_t^i} = & -\frac{c_t^i}{n_t^i} dt + dr_t^B + \theta_t^{K,i} \left( dr_t^{K,i} \left( l_t^i \right) - dr_t^B \right) \\ & + \theta_t^{E,i} \left( dr_t^{E,i} - dr_t^B \right) + \theta_t^{\bar{E},i} \left( d\bar{r}_t^E - dr_t^B \right). \end{aligned} \quad (4)$$

The household chooses consumption  $c_t^i$ , real investment  $l_t^i$ , and the portfolio shares  $\theta_t^{k,i}$ ,  $\theta_t^{E,i}$ , and  $\theta_t^{\bar{E},i}$  in capital, own outside equity and the diversified equity portfolio,

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<sup>7</sup>The own outside equity share  $\theta_t^{E,i}$  is negative as this asset is issued by the household.

respectively, to maximize utility  $V_0^i$  subject to (4) and the skin-in-the-game constraint

$$-\theta_t^{E,i} \leq (1 - \bar{\chi})\theta_t^{K,i}. \quad (5)$$

The HJB equation for this problem is (using the returns expressions from the previous paragraph)

$$\begin{aligned} & \rho V_t(n^i) - \partial_t V_t(n^i) \\ &= \max_{c^i, \theta^{K,i}, \theta^{E,i}, \theta^{\bar{E},i}} \left\{ \log c^i + V_t'(n^i) \left[ -c^i + n^i \left( \frac{\mathbb{E}_t[dr_t^B]}{dt} + \theta^{K,i} \left( \frac{\mathbb{E}_t[d\hat{r}_t^{K,i}(i)]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) \right. \right. \right. \\ & \quad \left. \left. \left. + \theta^{E,i} \left( \frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) + \theta^{\bar{E},i} \left( \frac{\mathbb{E}_t[dr_t^E]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) \right] \right. \\ & \quad \left. + \frac{1}{2} V_t''(n^i) (n^i)^2 \left( \left( \sigma_t^{q,B} - (\theta^{K,i} + \theta^{E,i} + \theta^{\bar{E},i}) \frac{\sigma_t^\theta}{1 - \theta_t} \right)^2 + (\theta^{K,i} + \theta^{E,i})^2 \tilde{\sigma}_t^2 \right) \right\}, \end{aligned}$$

where we have used  $\sigma_t^{q,K} - \sigma_t^{q,B} = \frac{\sigma_t^\theta}{1 - \theta_t}$ . As this is a standard portfolio choice problem, we conjecture a functional form  $V_t(n^i) = \alpha_t + \frac{1}{\rho} \log n^i$  for the value function,<sup>8</sup> where  $\alpha_t$  depends on (aggregate) investment opportunities, but not on individual net worth  $n^i$ .

Substituting this into the HJB and taking first-order conditions with respect to  $c^i$  and  $l^i$  yields the two equations

$$\begin{aligned} c_t^i &= \rho n_t^i, \\ \frac{d}{dt} \frac{\mathbb{E}_t[d\hat{r}_t^{K,i}(l)]}{dt} \Big|_{l=l_t^i} &= 0. \end{aligned}$$

The first condition is the familiar permanent income consumption equation for log preferences, the second condition reduces to a standard Tobin's  $q$  condition when combining it with the explicit formula for  $d\hat{r}_t^{K,i}(l)$  given above:<sup>9</sup>

$$q_t^K = \frac{1}{\Phi'(l_t^i)}.$$

<sup>8</sup>The verification argument is entirely standard, see e.g. Brunnermeier et al. (2020), Appendix A.2 for a proof.

<sup>9</sup>In particular, this equation implies that  $l_t^i = l_t$ .

Using the functional form  $\Phi(\iota) = \frac{1}{\phi} \log(1 + \phi\iota)$  and goods market clearing (2), the first two equations aggregated across agents imply

$$\begin{aligned}\iota_t &= \frac{(1 - \vartheta_t)(a_t - \mathfrak{g}_t) - \rho}{1 - \vartheta_t + \phi\rho}, \\ q_t^B &= \vartheta_t \frac{1 + \phi(a_t - \mathfrak{g}_t)}{1 - \vartheta_t + \phi\rho}, \\ q_t^K &= (1 - \vartheta_t) \frac{1 + \phi(a_t - \mathfrak{g}_t)}{1 - \vartheta_t + \phi\rho},\end{aligned}$$

which determines the equilibrium uniquely up to the nominal wealth share  $\vartheta_t$ .

$\vartheta_t$ , in turn, is determined by agents' portfolio choice. Taking the first-order condition in the HJB with respect to the three portfolio shares  $\theta_t^{K,i}$ ,  $\theta_t^{E,i}$ , and  $\theta_t^{\bar{E},i}$  yields three Merton portfolio choice equations

$$\begin{aligned}\frac{\mathbb{E}_t \left[ \frac{dr_t^{K,i}(i)}{dt} \right] - \frac{\mathbb{E}_t \left[ dr_t^B \right]}{dt}}{dt} &= \left( \sigma_t^{q,B} - \frac{\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}}{1 - \vartheta_t} \sigma_t^\vartheta \right) \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + (\theta_t^{K,i} + \theta_t^{E,i}) \tilde{\sigma}_t^2 - \lambda_t^i (1 - \bar{\chi}), \\ \frac{\mathbb{E}_t \left[ \frac{dr_t^{E,i}}{dt} \right] - \frac{\mathbb{E}_t \left[ dr_t^B \right]}{dt}}{dt} &= \left( \sigma_t^{q,B} - \frac{\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}}{1 - \vartheta_t} \sigma_t^\vartheta \right) \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + (\theta_t^{K,i} + \theta_t^{E,i}) \tilde{\sigma}_t^2 - \lambda_t^i, \\ \frac{\mathbb{E}_t \left[ \frac{d\bar{r}_t^E}{dt} \right] - \frac{\mathbb{E}_t \left[ dr_t^B \right]}{dt}}{dt} &= \left( \sigma_t^{q,B} - \frac{\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}}{1 - \vartheta_t} \sigma_t^\vartheta \right) \frac{\sigma_t^\vartheta}{1 - \vartheta_t}.\end{aligned}$$

Here,  $\lambda_t^i$  denotes the Lagrange multiplier on the constraint (5). Combining the last two equations and using  $\frac{\mathbb{E}_t \left[ \frac{d\bar{r}_t^E}{dt} \right]}{dt} = \frac{\mathbb{E}_t \left[ \frac{dr_t^{E,i}}{dt} \right]}{dt}$  in equilibrium, we obtain a simple characterization of  $\lambda_t^i$ :

$$\lambda_t^i = \left( \theta_t^{K,i} + \theta_t^{OE,i} \right) \tilde{\sigma}_t^2.$$

In particular,  $\lambda_t^i$  is always positive and thus the constraint (5) must be binding – households issue the maximum possible amount of outside equity. Then,  $\theta_t^{K,i} + \theta_t^{E,i} = \theta_t^{K,i} \bar{\chi}$ . Substituting this, the expression for  $\lambda_t^i$  and the return expressions into the first of the three portfolio choice conditions yields

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \check{\mu}_t^B}{1 - \vartheta_t} + \frac{(\sigma_t^{q,B} - \sigma_t^\vartheta) \sigma_t^\vartheta}{1 - \vartheta_t} = \left( \sigma_t^{q,B} - \frac{\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}}{1 - \vartheta_t} \sigma_t^\vartheta \right) \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \theta_t^{K,i} \bar{\chi}^2 \tilde{\sigma}_t^2.$$

The fact that all households choose the same portfolio shares and equity market clearing immediately imply  $\theta_t^{E,i} = -\theta_t^{\bar{E},i}$ . Furthermore, bond market clearing then requires  $1 - \theta_t^{K,i} = \vartheta_t$ . Substituting these clearing conditions and goods market clearing into the previous equation and solving for  $\mu_t^\vartheta$  gives us a condition for  $\vartheta_t$ :

$$\mu_t^\vartheta = \rho + \check{\mu}_t^B - (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2.$$

This is a backward equation for  $\vartheta_t$  that has been derived under the assumption that bonds have a positive value ( $\vartheta_t > 0$ ). In particular, in these cases multiplying the equation by  $\vartheta_t$  represents an equivalence transformation. Furthermore, if  $\vartheta_t = 0$ , then by no arbitrage, agents must expect also  $d\vartheta_t = 0$ ; otherwise, they could earn an infinite risk-free return from investing into bonds. Consequently, the backward stochastic differential equation (BSDE)

$$\mathbb{E}_t [d\vartheta_t] = \left( \rho + \check{\mu}_t^B - (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \right) \vartheta_t dt \quad (6)$$

must hold along any equilibrium path, regardless of whether bonds have positive value or not.

Together with a specification for the evolution of the exogenous states  $\tilde{\sigma}_t$ ,  $a_t$ , and  $g_t$  and for policy  $\check{\mu}_t^B$ , equation (6) determines the equilibrium process for  $\vartheta_t$ .

### 2.3 Safe Asset Debt Valuation Equation: Two Perspectives

Pricing government debt can be done from two different perspectives. First, the individual perspective recognizes that individual citizens do not intend to buy and hold the government bond, but plan to retrade it whenever they face a shock. After a negative shock, they raise cash flow by selling the bond, while after a positive shock they buy additional bonds. The cash flow stream associated with this optimal trading strategy is stochastic. Instead of pricing the government bond directly, it is insightful to “price” the cash flows from the optimal stochastic trading strategy and then aggregate over all individuals. Second, the aggregate perspective prices government debt from the government perspective. Hence, in a setting without aggregate risk the bond is risk-free and future payoffs are discounted at the risk-free rate. In a setting with aggregate risk, only the aggregate component of the stochastic discount factor enters the debt

valuation equation. Note also dynamic programming implies that the transversality condition has to hold only from the individual perspective, for each citizen. Optimality does not imply a transversality condition from the aggregate perspective (where discounting happens at a lower effective rate).

**Individual Perspective.** We denote the individual SDF process of citizen  $i$  with  $\tilde{\zeta}_t^i$ . This process satisfies  $d\tilde{\zeta}_t^i/\tilde{\zeta}_t^i = -r_t^f dt - \zeta_t dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t^i$ , with a negative drift term equal to the risk-free rate and aggregate and idiosyncratic price of risk terms,  $\zeta_t$ ,  $\tilde{\zeta}_t^i$  respectively.<sup>10</sup> Also, let  $\eta_t^i := n_t^i/N_t$  be citizens  $i$ 's net worth share. The individual perspective asset pricing yields our main valuation equation,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[ \int_0^\infty \left( \int \tilde{\zeta}_t^i \eta_t^i di \right) s_t K_t dt \right] + \mathbb{E} \left[ \int_0^\infty \left( \int \tilde{\zeta}_t^i \eta_t^i di \right) (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right]. \quad (7)$$

The real value of all outstanding public debt  $\frac{\mathcal{B}_0}{\mathcal{P}_0}$  consists of two terms, the discounted value of future primary surpluses,  $s_t K_t := (\tau_t a - \mathfrak{g}_t) K_t$ , plus the discounted value of future service flows,  $(1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}$ . The safe asset service flow is due to partial insurance, which increases in the value of public debt, and the amount of idiosyncratic risk the citizen is exposed to, which in turn depends on his portfolio share on physical capital  $(1 - \vartheta_t)$  and  $\tilde{\sigma}_t$ . Importantly, the stochastic discount factor in this equation is a net-worth-weighted average of individual stochastic discount factors. Since a single citizen's individual net worth weight  $\eta_t^i$  co-moves negatively with his SDF  $\tilde{\zeta}_t^i$ , the discount factor is lower (discount rate is higher) than the usual unweighted average discount factor (used in aggregate perspective below).

To obtain valuation equation (7) we start valuing citizen  $i$ 's bond portfolio at time

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<sup>10</sup>In integral form the individual SDF is

$$\tilde{\zeta}_t^i = \underbrace{\exp \left( - \int_0^t r_\tau^f d\tau \right)}_{\text{time discounting}} \cdot \underbrace{\exp \left( - \int_0^t \zeta_\tau dZ_\tau - \frac{1}{2} \int_0^t \zeta_\tau^2 d\tau \right)}_{\text{aggregate risk}} \cdot \underbrace{\exp \left( - \int_0^t \tilde{\zeta}_\tau d\tilde{Z}_\tau - \frac{1}{2} \int_0^t \tilde{\zeta}_\tau^2 d\tau \right)}_{\text{idiosyncratic risk}},$$

where the second and third factors are martingales.

$t = 0$ :<sup>11</sup>

$$n_0^{b,i} = \mathbb{E} \left[ \int_0^\infty \bar{\zeta}_t^i \left( \underbrace{c_t^i}_{\text{consumption}} - \underbrace{k_t^i (a - \iota_t - \tau_t a_t)}_{\text{production net of investment and taxes}} - \underbrace{q_t^K k_t^i (1 - \vartheta_t) \bar{\chi} \bar{\sigma}_t \bar{\sigma}_t^{\Delta,i}}_{\text{cash flows from trading capital}} \right) dt \right], \quad (8)$$

where  $n_0^{b,i} := \theta_0^i n_0^i$  is the initial bond wealth of agent  $i$ . This equation says that the initial bond wealth of the household must equal the discounted value of future consumption in excess of the citizen  $i$ 's own production net of reinvestment and tax payments and in excess of trading expenses for purchasing new capital (these can be negative if capital is sold). In this model, capital trading only happens in response to idiosyncratic shocks,  $d\Delta_t^{k,i} = \bar{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$ .<sup>12</sup> In the appendix we show that  $\bar{\sigma}_t^{\Delta,i} = -\vartheta_t \bar{\chi} \bar{\sigma}_t$ .

Next, replacing individual with scaled aggregate variables,  $c_t^i = \eta_t^i C_t$  and  $k_t^i = \eta_t^i K_t$ , one obtains

$$n_0^{b,i} = \mathbb{E} \left[ \int_0^\infty \bar{\zeta}_t^i \eta_t^i \left( \tau_t a_t K_t + C_t - (a - \iota_t) K_t + (1 - \vartheta_t) \bar{\chi}^2 \bar{\sigma}_t^2 \vartheta_t q_t^K K_t \right) dt \right].$$

Including the aggregate resource constraint (2),  $C_t - (a - \iota_t) K_t = \mathfrak{g}_t K_t$ , the fact that,  $n_0^{b,i} = \vartheta_0 n_0^i = \eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0}$ , and,  $\vartheta_t q_t^K K_t = (1 - \vartheta_t) \frac{\mathcal{B}_t}{\mathcal{P}_t}$  leads to

$$\eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[ \int_0^\infty \bar{\zeta}_t^i \eta_t^i s_t K_t dt \right] + \mathbb{E} \left[ \int_0^\infty \bar{\zeta}_t^i \eta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right]. \quad (9)$$

Finally, integrating over individuals  $i$  yields equation (7).

**Aggregate Perspective.** From the aggregate perspective, individual uninsurable risk does not enter the valuation equation directly. Indeed, absent aggregate shocks (including inflation shocks), the government bond is a risk-free asset. Viewed from this perspective we obtain a different, an aggregate, discount factor process,  $d\bar{\zeta}_t / \bar{\zeta}_t = -r_t^f dt - \zeta_t dZ_t$ .<sup>13</sup> Absent aggregate risk the discount factor is simply  $\bar{\zeta}_t = \exp(-\int_0^t r_\tau^f d\tau)$ .

<sup>11</sup>This equation is an immediate consequence of the agent's intertemporal budget constraint. In particular, a transversality condition always ensures that there is no additional nonvanishing terminal wealth term.

<sup>12</sup>While agents expect to make as many purchases as sales in the future, so that the expected cash flows from trading are zero, there is nevertheless a trading term in equation (8) that reflects the covariance between cash flows from trading and individual marginal utility.

<sup>13</sup>The aggregate discount factor is the projection of any individual citizen's SDF onto a common filtration generated by the aggregate Brownian  $\{Z_t\}_{t=0}^\infty$ . Put differently,  $\bar{\zeta}_t := \mathbb{E} \left[ \zeta_t^i \mid Z_\tau : \tau \leq t \right]$ , takes

The government debt valuation equation at  $t = 0$  is

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T \rightarrow \infty} \left( \mathbb{E} \left[ \int_0^T \bar{\zeta}_t s_t K_t dt \right] + \mathbb{E} \left[ \bar{\zeta}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right), \quad (10)$$

consisting of two terms: a discounted stream of primary surpluses plus (the limit of) a discounted terminal value. The latter can be positive even in the limit, giving rise to a possible bubble on government debt.<sup>14</sup> The reason is that in our model no private citizen's transversality condition necessary implies  $\mathbb{E} \left[ \bar{\zeta}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \rightarrow 0$  because agents do not buy and hold a fixed fraction of the government debt stock but constantly trade bonds. If the discount factor is small enough so that the terminal condition does converge to zero, we obtain the traditional debt valuation equation that says that the value of debt must equal the present value of primary surpluses.

To obtain equation (10), we start by using  $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$  to rewrite the government flow budget constraint (1) as

$$- (d\mathcal{B}_t - i_t \mathcal{B}_t dt) = \mathcal{P}_t \underbrace{(\tau a_t - g_t)}_{=s_t} K_t dt,$$

where  $s_t$  denotes again the government primary surplus normalized by the aggregate capital stock.

We now multiply both sides by the nominal SDF  $\zeta_t^i / \mathcal{P}_t$  of agent  $i$  and use Ito's product rule to replace  $\zeta_t^i / \mathcal{P}_t d\mathcal{B}_t$  with  $d \left( \zeta_t^i / \mathcal{P}_t \mathcal{B}_t \right) - \mathcal{B}_t d \left( \zeta_t^i / \mathcal{P}_t \right)$ :<sup>15</sup>

$$-d \left( \zeta_t^i \mathcal{B}_t / \mathcal{P}_t \right) + \mathcal{B}_t \left( d \left( \zeta_t^i / \mathcal{P}_t \right) + i_t \zeta_t^i / \mathcal{P}_t dt \right) = \zeta_t^i s_t K_t dt.$$

Integrating this equation from  $t = 0$  to  $t = T$ , taking expectations, and solving for  $\zeta_0^i \mathcal{B}_0 / \mathcal{P}_0$  yields

$$\zeta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[ \int_0^T \zeta_t^i s_t K_t dt \right] - \mathbb{E} \left[ \int_0^T \mathcal{B}_t \left( d \left( \zeta_t^i / \mathcal{P}_t \right) + i_t \zeta_t^i / \mathcal{P}_t dt \right) \right] + \mathbb{E} \left[ \zeta_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T} \right]. \quad (11)$$

conditional expectations with respect to the history of aggregate shocks  $dZ_\tau$  up to time  $t$  but without any knowledge of idiosyncratic shocks. Equivalently,  $\bar{\zeta}_t = \int \zeta_t^i di$  is the unweighted average of individual SDFs.

<sup>14</sup>The bubble term on government debt is discussed in detail in Brunnermeier et al. (2020).

<sup>15</sup>There is no quadratic covariation term because  $d\mathcal{B}_t$  is absolutely continuous.

Equation (11) is simply an accounting identity, the government flow budget constraint (1) multiplied with the discounting process  $\zeta_t^i/\mathcal{P}_t$ . We now add economic content by noting that the individual SDF  $\zeta_t^i$  must price the bond because agent  $i$  is marginal in the bond market. This implies that the associated nominal SDF  $\zeta_t^i/\mathcal{P}_t$  must decay on average at the nominal market interest rate, so that the second term in equation (11) vanishes. In addition, we can replace the individual SDF  $\zeta_t^i$  with the average SDF  $\bar{\zeta}_t$  because equation (11) holds for all individuals  $i$  and  $s_t K_t$  and  $\mathcal{B}_T/\mathcal{P}_T$  are free of idiosyncratic risk. When taking the limit  $T \rightarrow \infty$ , we obtain equation (10)

**Comparison of the Two Approaches** The SDFs used in equations (7) and (10) are both free of idiosyncratic risk and imply the same aggregate risk premium, but they differ with respect to their average rate of decay, the “risk-free rate” they imply. While the average SDF  $\bar{\zeta}$  decays at rate  $r_t^f$  and is thus a “proper” SDF in this model that prices all assets free of idiosyncratic risk, this is not true for the weighted average SDF  $\int \zeta_t^i \eta_t^i di$ . The latter decays at a rate  $r_t^f + \bar{\zeta}_t \bar{\sigma}_t^n$ , where  $\bar{\sigma}_t^n$  is the idiosyncratic net worth volatility of agents (which is identical for all agents in equilibrium). The weighted average SDF  $\int \zeta_t^i \eta_t^i di$  therefore discounts safe cash flows at a higher rate than the risk-free rate.

These considerations imply that only equation (10) is a standard asset pricing condition, a discounted present value formula using a SDF that prices all assets (at least those free of idiosyncratic risk). However, (10) can have a nonzero bubble term. Unfortunately, it can even happen that both the bubble term and the present value of primary surpluses are infinite with opposite sign, yet their sum still converges as  $T \rightarrow \infty$ . In contrast, the integrals in equation (7) are always well-defined and finite. Working with equation (7) can therefore be more informative, even though it uses a SDF that does not price the government debt.

## 2.4 Closed-Form Steady State and Gordon Growth Formulas

In this section, we assume that productivity  $a$ , idiosyncratic risk  $\tilde{\sigma}$ , and government spending per unit of capital  $g$  are constant. We also restrict attention to government policies that hold taxes  $\tau$  constant over time and characterize steady-state equilibria with constant  $q^B$  and  $q^K$  and a positive value of government bonds,  $q^B > 0$ . These assumptions immediately imply that also  $\vartheta$  and  $\tilde{\mu}^B$  must be constant in such a steady state.



Any such equilibrium must thus solve equation (6) with  $d\vartheta_t = 0$ . The right-hand side is a third-order polynomial, so there are three solutions to this equation,  $\vartheta = 0$ ,  $\vartheta = \frac{\bar{\chi}\bar{\sigma} + \sqrt{\rho + \check{\mu}^B}}{\bar{\chi}\bar{\sigma}}$ , and  $\vartheta = \frac{\bar{\chi}\bar{\sigma} - \sqrt{\rho + \check{\mu}^B}}{\bar{\chi}\bar{\sigma}}$ . Among these solutions, only the third can be consistent with  $q^B, q^K > 0$  and thus a valid steady state equilibrium in which bonds have a positive value.<sup>16</sup> It is consistent with such an equilibrium if in addition the condition

$$\bar{\chi}\bar{\sigma} \geq \sqrt{\rho + \check{\mu}^B}$$

is satisfied. Effectively, this inequality imposes a constraint on bond growth in excess of interest payments  $\check{\mu}^B$  for the private sector to remain willing to hold government bonds.

In this case, investment is

$$\iota = \frac{\sqrt{\rho + \check{\mu}^B} (a - g) - \rho \bar{\chi}\bar{\sigma}}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\bar{\sigma}}$$

and the (scaled) real asset values are

$$q^B = \frac{(\bar{\chi}\bar{\sigma} - \sqrt{\rho + \check{\mu}^B}) (1 + \phi (a - g))}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\bar{\sigma}}, \quad q^K = \frac{\sqrt{\rho + \check{\mu}^B} (1 + \phi (a - g))}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\bar{\sigma}}.$$

While these expressions have the advantage of being an explicit model solution in terms of parameters, for interpretation it is helpful to write the last two equations as Gordon growth formulas

$$q^B = \frac{s + (1 - \vartheta)^2 \bar{\chi}^2 \bar{\sigma}^2 q^B}{\mathbb{E}[dr^n]/dt - g}, \quad q^K = \frac{(1 - \tau)a - \iota}{\mathbb{E}[dr^K]/dt - g}.$$

Here, the second equation follows from the fact that the price of a single unit of capital must be the present value of cash flows generated by that unit of capital. The current period cash flow is production net of taxes and reinvestment,  $(1 - \tau)a - \iota$ , and the expected growth rate of these cash flows is the economy's growth rate  $g := \Phi(\iota) - \delta$ .<sup>17</sup> Because capital is risky, expected cash flows must be discounted at the expected

<sup>16</sup>The second solution never corresponds to a valid equilibrium, while the first is only consistent with equilibrium if government primary surpluses are zero, see Brunnermeier et al. (2020) for details.

<sup>17</sup> $g$  is both the growth rate of output and of the aggregate capital stock.

return on capital  $\mathbb{E}[dr^K]/dt$ , which includes a risk premium for idiosyncratic risk.

The first equation is a consequence of equation (7), the individual perspective on government debt valuation. Per unit of aggregate capital in the economy, the “cash flow” on bonds consists of the surplus-capital ratio  $s$  and the service flow  $(1 - \vartheta)^2 \bar{\chi} \tilde{\sigma}^2 q^B$  from trading bonds to self-insure against idiosyncratic risk. Both types of cash flows grow on average at the economy’s growth rate, but are risky from the individual’s perspective. The required discount rate is therefore  $\mathbb{E}[dr^n]/dt$ , where  $dr^n = \vartheta dr^B + (1 - \vartheta)dr^K$  denotes the return on the agents (net worth) portfolio, because the idiosyncratic risk of net worth is precisely the residual idiosyncratic risk that the agent has to bear after optimal re-trading of bonds.<sup>18</sup>

### 3 Counter-cyclical Safe Asset and 2 Betas

#### 3.1 Model Setup with Stochastic Idiosyncratic Risk and Recursive Utility

We introduce aggregate risk as shocks to idiosyncratic risk  $\tilde{\sigma}_t$ . We interpret periods of high idiosyncratic risk as recessions and want them to be associated with lower consumption and higher marginal utility. Rather than microfounding this relationship explicitly, we simply impose exogenous relationships  $a_t = a(\tilde{\sigma}_t)$  and  $g = g(\tilde{\sigma}_t)$  that are consistent with the desired correlation structure.<sup>19</sup>

For idiosyncratic risk  $\tilde{\sigma}_t$ , we specify a Heston (1993) model of stochastic volatility, i.e. we assume that the idiosyncratic variance  $\tilde{\sigma}_t^2$  follows a Cox–Ingersoll–Ross process (Cox et al., 1985) process,

$$d\tilde{\sigma}_t^2 = -\psi \left( \tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2 \right) dt - \sigma \tilde{\sigma}_t dZ_t$$

with parameters  $\psi, \sigma, \tilde{\sigma}^0 > 0$ .

To ensure that  $C_t/K_t$  is strictly decreasing in  $\tilde{\sigma}_t$ , we do not directly specify functions

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<sup>18</sup>More formally,  $\eta_t^i = n_t^i/N_t$ , so that the relative risk in  $n_t^i$  is the same as the relative risk in  $\eta_t^i$  and the latter matters for discounting when using the weighted-average discount factor  $\int \zeta_t^i \eta_t^i di$ .

<sup>19</sup>For models similar to ours in which output and consumption naturally react negatively to risk shocks, see DiTella and Hall (2020) and Li and Merkel (2020).

$a(\tilde{\sigma})$  and  $g(\tilde{\sigma})$ , but instead impose for the endogenous consumption-capital ratio the equation

$$C/K(\tilde{\sigma}_t) := \alpha_0 - \alpha_1 \tilde{\sigma}_t$$

for some parameters  $\alpha_0, \alpha_1 > 0$ . Because equation (6) implies that  $\vartheta_t$  is determined independently of the processes for  $a_t$  and  $g_t$ ,<sup>20</sup> we can first solve for the solution function  $\vartheta(\tilde{\sigma})$  using just the specification for the  $\tilde{\sigma}_t$  process and then invert the formula  $C_t/K_t = \rho \frac{1+\phi(a_t-g_t)}{1-\vartheta_t+\phi\rho}$  to back out the required function  $a - g$  to obtain the desired consumption-capital ratio in equilibrium.<sup>21</sup>

For government policy, we assume that debt growth net of interest payments satisfies a linear relationship

$$\check{\mu}_t^B = -\nu_0 + \nu_1 \tilde{\sigma}_t \quad (12)$$

with parameters  $\nu_0, \nu_1 > 0$ . Provided  $\nu_1$  is sufficiently large, this implies that surpluses  $s_t = -\check{\mu}_t^B q_t^B$  are positive for low idiosyncratic risk (in expansions) and negative for high idiosyncratic risk (in recessions). Primary surpluses therefore correlate negatively with marginal utility and any agent in the economy would require a positive risk premium for holding a (hypothetical) claim to primary surpluses.

Finally, in order for our model to generate quantitatively realistic aggregate risk premia, we work in this section with a slightly more general version of the model. Specifically, we replace logarithmic preferences of households with stochastic differential utility (Duffie and Epstein, 1992) with unit elasticity of intertemporal substitution and arbitrary relative risk aversion  $\gamma > 0$ : household  $i$  maximizes  $V_0^i$ , where  $V_t^i$  is recursively defined by

$$V_t^i = E_t \left[ \int_t^\infty (1 - \gamma) \rho V_s^i \left( \log(c_s^i) - \frac{1}{1 - \gamma} \log \left( (1 - \gamma) V_s^i \right) \right) ds \right].$$

In the special case  $\gamma = 1$ , this specification collapses to our baseline specification with logarithmic utility discussed in Section 2.<sup>22</sup>

<sup>20</sup>This is also true for the extension with stochastic differential utility introduced below.

<sup>21</sup>The processes  $a$  and  $g$  are not individually relevant for anything of interest here, just their difference  $a - g$  is.

<sup>22</sup>Qualitatively, the two models behave identically. However, with  $\gamma = 1$ , the model does not generate a sufficiently large aggregate price of risk to capture the empirically observable equity premium.

## 3.2 Calibration

We calibrate our model such that, when we feed in a quantitatively realistic process for idiosyncratic risk, the model generates variations in output, consumption, and surpluses and aggregate risk premia that are broadly consistent with US data. For our mapping from the model to the data, one time period in the model corresponds to one year.

We take the parameters  $\tilde{\sigma}^0$ ,  $\psi$ ,  $\sigma$  for the exogenous  $\tilde{\sigma}_t$  process from Merkel (2020), who reinterprets the idiosyncratic capital shocks as idiosyncratic TFP shocks and chooses the exogenous process parameters to match the evidence on establishment-level idiosyncratic TFP shocks reported by Bloom et al. (2018). We set the fraction  $\bar{\chi}$  of idiosyncratic risk that must be retained by insiders to one half in line with the evidence on the contribution of private equity to the wealth of US investors reported by Angeletos (2007). We set the capital adjustment cost parameter  $\phi$  to 6 (currently uncalibrated).

We choose the remaining six parameters  $\gamma, \rho, \alpha_0, \alpha_1, \nu_0, \nu_1$  such that the model generates values for the volatilities of output, consumption and the surplus-output ratio, the average consumption-output, surplus-output, capital-output and debt-output ratios as well as the equity premium and equity sharpe ratio that are broadly in line with the empirical evidence.

Table 1: Parameter Choice

parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.29
$\psi$	$\tilde{\sigma}_t^2$ mean reversion	0.15
$\sigma$	$\tilde{\sigma}_t^2$ volatility	0.037
$\bar{\chi}$	undiversifiable idio. risk	0.5
$\phi$	capital adjustment cost	6
$\gamma$	risk aversion	7.5
$\rho$	time preference	0.17
$\alpha_0$	$C/K$ intercept	0.59
$\alpha_1$	negative of $C/K$ slope	0.2
$\nu_0$	$\check{\mu}^B$ intercept	0.085
$\nu_1$	$\check{\mu}^B$ slope	0.25

Table 1 summarizes our parameter choice and Table 2 summarizes the quantitative model fit. We report data moments both for our full sample (1966–2019) and for the

Table 2: Quantitative Model Fit

symbol	moment description	model	data	
			full sample (1966–2019)	post 1985
$\sigma(Y)$	output volatility	0.019	0.014	0.010
$\sigma(C)$	consumption volatility	0.010	0.008	0.007
$\sigma(S/Y)$	surplus volatility	0.004	0.009	0.011
$\rho(Y, C)$	correlation of output and consumption	0.977	0.826	0.83
$\rho(Y, S/Y)$	correlation of output and surpluses	0.927	0.471	0.710
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.667	0.615	0.614
$\mathbb{E}[S/Y]$	average surplus-output ratio	0.007	0.007	0.004
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.206	$\approx 3$	
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.672	0.578	0.714
$\mathbb{E}[dr^E - dr^B]$	average equity premium	5.6%	$\approx 6.4\%$	
$\frac{\mathbb{E}[dr^E - dr^B]}{\sigma(dr^E - dr^B)}$	equity sharpe ratio	0.436	$\approx 0.5$	

**Notes:**  $\sigma(x)$  denotes the standard deviation of  $x$  and  $\rho(x, y)$  denotes the correlation of  $x$  and  $y$ , both at a quarterly frequency. Inputs  $x$  and  $y$  are HP-filtered with smoothing parameter 1600. For  $x, y \in \{Y, C\}$ , we take logarithms before filtering.  $\mathbb{E}[x]$  denotes expectations over the ergodic model distribution, inputs  $x$  are *not* HP-filtered.  $Y$ : (aggregate) output,  $C$ : consumption,  $S$ : primary surplus,  $q^K, q^B, dr^B, dr^E$  are defined as in Section 2.

post-1985 period, as only during the latter US government debt has been a negative- $\beta$  asset.<sup>23</sup>

The model generates output and consumption volatility that is slightly higher than but of similar magnitude as in the data. Government surpluses are slightly less volatile in our model, but they are also more correlated with output due to the fact that everything in our model is driven by a single shock. In total, the component of surplus variation that is systematically comoving with output is thus approximately as volatile as in the data.<sup>24</sup>

Table 2 also shows that our model does a good job at matching a number of important first moments, including the equity premium (and the equity sharpe ratio). The latter is particularly important because it verifies that our model is capable of generating realistic aggregate risk premia.

<sup>23</sup>This is mainly due to the stagflation episode of the 1970s. Our simple model with a single state variable cannot account for occasional stagflation episodes.

<sup>24</sup>It is this part of surplus variation that ultimately matters for asset pricing.

### 3.3 Analyzing the Two Asset Pricing Terms Separately

We now consider the two terms in the government debt valuation equation derived from the individual perspective (equation (7)). Figure 1 plots the two present values<sup>25</sup>

$$q^{B,cf}(\tilde{\sigma}) := \mathbb{E} \left[ \int_0^\infty \left( \int \xi_t^i \eta_t^i di \right) s_t K_t dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0 \right] / K_0$$

$$q^{B,sf}(\tilde{\sigma}) := \mathbb{E} \left[ \int_0^\infty \left( \int \xi_t^i \eta_t^i di \right) (1 - \vartheta_t)^2 \gamma \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{B_t}{P_t} dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0 \right] / K_0$$

for our calibrated model. The blue solid line shows the present value of future primary surpluses (cash flows)  $q^{B,cf}$  as a function of the single state variable  $\tilde{\sigma}$ . This value is strictly decreasing in idiosyncratic risk and has a low – and sometimes negative – value. Comparing the present value of surpluses  $q^{B,cf}K$  in our model to the market value of government debt  $q^B K$ , which is represented by the black dashed line in Figure 1, reveals a large gap  $(q^B - q^{B,cf})K$ , a “debt valuation puzzle”. In addition, when compared with the present value of surpluses  $q^{B,cf}K$ , the total value of government debt  $q^B K$  has also the opposite correlation with the aggregate state. Yet, there is no puzzle from the perspective of our model: government debt is a safe asset valued for its service flow from re-trading which is represented by the component  $q^{B,sf}(\tilde{\sigma})$ . As the red solid line in Figure 1 shows, this value is positive, large and positively correlated with  $\tilde{\sigma}_t$ . This additional component dominates the overall dynamics of the value of government debt and is the reason that  $q^B$  appreciates in bad times despite the simultaneous drop in  $q^{B,cf}$ . That  $q^{B,sf}$  must be positively correlated with  $\tilde{\sigma}$  can also be seen from the present value equation: one can show that for our policy specification residual net worth risk  $(1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t$  must be strictly increasing in  $\tilde{\sigma}_t$ , so that an increase in idiosyncratic risk increases the value of insurance service flows from re-trading.<sup>26</sup>

The correlation structure apparent in Figure 1 implies that, if the two claims  $q^{B,cf}$  and  $q^{B,sf}$  could be traded separately, the cash flow claim would be a high- $\beta$  asset,

<sup>25</sup>Relative to equation (7), here an additional factor  $\gamma$  appears because we no longer assume logarithmic preferences.

<sup>26</sup>This is not an entirely rigorous argument as it ignores changes in the discount rate. The effective discount rate in the weighted-average SDF  $\int \xi_t^i \eta_t^i di$  can both increase or decrease with the aggregate state  $x_t$  depending on whether the *aggregate* risk premium increases or decreases. Note however, that the level of idiosyncratic risk does not directly matter for the effective discount rate because the risk premium on idiosyncratic risk exactly offsets the lower risk-free rate due to a precautionary motive.

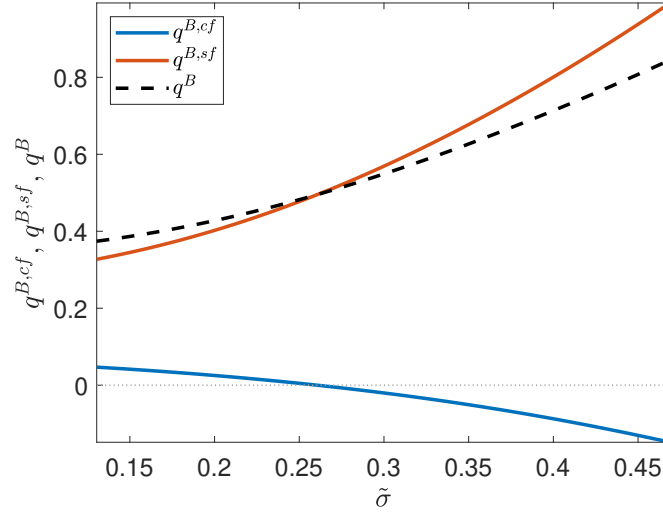


Figure 1: Decomposition of the value of government debt as a function of idiosyncratic risk  $\tilde{\sigma}$ . The blue solid line shows the present value of primary surpluses ( $q^{B,cf}$ ), the red solid line the present value of service flows ( $q^{B,sf}$ ) and the black dashed line the total value of government debt ( $q^B$ ), all normalized by the capital stock.

while the service flow claim would be a negative- $\beta$  asset. The presence of this second, negative- $\beta$  component makes government debt as a whole a negative  $\beta$  asset. Government debt emerges as a “good friend” also with respect to aggregate shocks. Figure 2 depicts this explicitly by plotting (weighted) conditional betas for the two hypothetical assets.<sup>27</sup>

### 3.4 The Possibility of Insuring Bond Holders and Tax Payers at the Same Time

In our simple setting citizens are capital owners and bond holders. In this section, we conceptually separate each household into two sub-units, a capital owner and a government debt holder. Surprisingly, it is possible to follow a government policy that provides tax payers insurance against negative aggregate shocks and bond holders at the same time. By cutting taxes (or even granting subsidies) for capital owners in

<sup>27</sup>We define  $\beta_t^i = \sigma_t^i / \zeta_t$ , where  $i \in \{cf, sf\}$  and  $dr^i$  is the return on the respective component and  $\sigma_t^i$  is the aggregate risk loading of that return. This definition can be interpreted as  $\beta_t = -\frac{\text{cov}_t(d\zeta_t/\zeta_t, dr^i)}{\text{var}_t(d\zeta_t/\zeta_t)}$ , where  $d\zeta_t/\zeta_t$  is the SDF that discounts cash flows from  $t + dt$  to time  $t$ . In addition, we weight  $\beta^i$  by its share  $\omega^i := q^{B,i}/q^B$  of the total government debt claim.

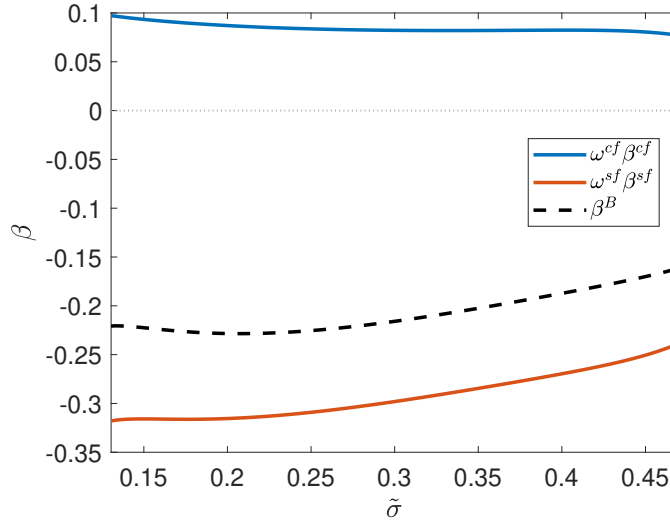


Figure 2: Conditional betas of hypothetical claims to the surplus and risk-sharing components of the government debt value

recessions, their tax burden is positively correlated with their income providing insurance to tax payers. At the same time, the safe asset premium rises in recessions, which provides insurance to government bond holders. Importantly, this finding in our incomplete market setting with a safe-asset bubble is in sharp contrast to traditional asset pricing in which either tax payers or government bond holders can insured, as pointed out in Jiang et al. (2020).

## 4 Implications for Equity Markets

The presence of idiosyncratic risk and government debt as a safe asset also has implications for equity markets. We explain in this section why the diversified equity portfolio does not emerge as a safe asset and how flight to safety can generate additional equity return volatility.

**Why Stocks Are not Safe Assets.** In our model, agents can hold a diversified stock portfolio. Like government bonds, this stock portfolio is free of idiosyncratic risk and thus allows agents to self-insure against idiosyncratic consumption fluctuations. However, unlike government bonds, stocks are poor aggregate risk hedges as they are ultimately claims to capital, which loses in value in recessions. This implies that stocks



are positive- $\beta$  assets in our model.

To understand why stock prices fall in times of high idiosyncratic risk, even though idiosyncratic equity risk can be diversified away, note that the marginal holder of capital in our model is always an insider who has to bear the increased idiosyncratic risk. As a consequence, when idiosyncratic risk goes up, so does the insider premium earned by the managing households, which is achieved by a reduction in the dividend that is paid to outside equity holders. This makes stock dividends more procyclical than production cash flows, so that stocks lose value precisely when idiosyncratic risk goes up.

When evaluating the diversified stock portfolio with regard to the two key characteristics of safe assets, the Good Friend Analogy and the Safe Asset Tautology, stocks fail to qualify as safe assets in the same way as government debt does. Stocks have the good friend characteristic only partially: stocks are valuable and liquid when an agent experiences a negative idiosyncratic shock, but due to their positive  $\beta$ , they are not in bad aggregate times. The positive  $\beta$  property and the absence of a safe asset bubble on stocks also means that stocks do not have a safe asset status in the sense of the Safe Asset Tautology, either.<sup>28</sup>

**Flight-to-safety Volatility.** While the focus of this paper is on government bonds, our model can match the empirical mean and volatility of the excess return on the stock market in excess of government bonds. The realistic sharpe ratio is clearly a feature of recursive preferences with a high risk aversion, but the ability of our simple model to generate large return volatility in the presence of realistic levels of output variation is quite remarkable and directly related to the existence of safe government bonds.

To gain intuition, let's abstract from the distinction between capital and outside equity<sup>29</sup> and for a moment also switch off both government spending  $G_t$  physical capital investments  $I_t$  by putting  $g = 0$  and considering the limit  $\phi \rightarrow \infty$ , so that  $Y_t = C_t$ . Then, aggregating individual households' intertemporal budget constraints yields the equation

$$q_t^K K_t + q_t^B K_t = \mathbb{E}_t \left[ \int_t^\infty \frac{\int \bar{\zeta}_s^i \eta_s^i di}{\int \bar{\zeta}_t^i \eta_t^i di} Y_s ds \right]. \quad (13)$$

<sup>28</sup>There may be alternative equilibria which feature bubbles on stocks. We defer the discussion of those to Section 6.

<sup>29</sup>As we have discussed previously in this section, a state-dependent insider premium will ensure that equity values and capital values move in lockstep despite the fact that idiosyncratic equity risk can be diversified away.

In a standard Lucas-type economy there would be no government debt with a positive net value,  $q_t^B = 0$ , and thus equation (13) would imply that the value of the capital stock equals the present value of future output. In other words, in a Lucas-type economy, pricing the aggregate equity claim is equivalent to pricing the aggregate output claim.<sup>30</sup> In the presence of realistic output volatility, a large volatility in capital valuations  $q_t^K K_t$  is hard to generate (and requires substantial time variation in the SDF  $\int \zeta_s^i \eta_s^i di$ ). In the presence of  $G_t, I_t \neq 0$ , the puzzle tends to become larger because consumption is smoother than output in the data.

In our model,  $q_t^B \neq 0$  and this suggests an additional explanation for the high observed stock market volatility. When idiosyncratic risk  $\tilde{\sigma}_t$  rises, there is a flight to safety that increases the value of bonds ( $q_t^B$ ) and lowers the value of capital ( $q_t^K$ ). Even in the absence of changes in the present value on the right-hand side of equation (13), this portfolio reallocation generates *flight-to-safety volatility* in capital valuations and thus in the stock market.

To get a sense how much flight-to-safety volatility matters quantitatively, we compare the excess stock return volatility in our model to the one generated by a version of the model without government debt (and primary surpluses set to zero). In that alternative version,  $q_t^B = 0$  at all times and thus flight-to-safety volatility disappears.<sup>31</sup> To make the comparison fair, we compute excess returns in this alternative model not in excess of the risk-free rate but in excess of a (zero net supply) asset that has the same negative  $\beta$  as government debt in our baseline model.<sup>32</sup> We find that the average (annualized) excess return volatility in the alternative model would be 2.9% as opposed to 12.9% in our baseline model. We can therefore conclude that flight-to-safety volatility accounts for more than three quarters of the overall excess return volatility in our framework.

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<sup>30</sup>Because the equation results from aggregating individual intertemporal budget constraints, the SDF used in this pricing equation is again the weighted-average SDF as in the individual perspective to government bond valuation, not any market SDF (i.e. a SDF that prices all tradeable assets). Of course, in most Lucas-type models there is no idiosyncratic risk so that the two coincide.

<sup>31</sup>Except for the elimination of primary surpluses ( $\nu_0 = \nu_1 = 0$ ) and the selection of the “non-monetary” equilibrium, we keep all other parameters as in our baseline model.

<sup>32</sup>Specifically, we take the  $\beta(\tilde{\sigma})$  function from the solution of the model with government debt and price a benchmark “bond” asset in the alternative model that has the return volatility  $\sigma_t^{r^B} = \beta(\tilde{\sigma}_t)\zeta_t$ , where  $\zeta_t$  is the (common) price of aggregate risk in all agents’ SDF in the alternative model.

## 5 Mining the Bubble: The Debt-Laffer Curve

The potential presence of a bubble in equation (10) in the aggregate perspective and the service flow term in equation (7) in the individual perspective suggest that the safe asset status of government debt represents a fiscal resource that the government can “mine” for revenue instead of taxation. Indeed, when the government chooses a permanently positive bond growth in excess of interest payments  $\check{\mu}^B$  (and thus permanently negative primary surpluses), the value of government debt may not collapse despite the negative present value of primary surpluses.

Our model therefore implies that the government may be able to finance government expenditures by “mining the bubble” without ever raising taxes for it. It can do so if undiversified idiosyncratic risk is sufficiently severe (high  $\bar{\chi}\bar{\sigma}$ ) such that even in the absence of positive surpluses government debt retains a positive value because of a bubble component. In steady state, this is the case under the condition  $\bar{\chi}\bar{\sigma} > \sqrt{\rho/\gamma}$ , which is equivalent to  $r^f \leq g$ .<sup>33</sup>

If this condition is satisfied, does the existence of a bubble imply that the government faces no budget constraint and can expand spending without limits? The answer is of course no as real resources are still finite and the real value of government debt reacts to the policy choice. Specifically, primary deficits per unit of capital are given by<sup>34</sup>

$$-s_t = \check{\mu}_t^B q_t^B.$$

The first factor,  $\check{\mu}_t^B$ , measures revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a higher  $\check{\mu}_t^B$  represents a tax on existing bond holders. The second factor,  $q_t^B$ , is the tax base, the real value of existing debt (per unit of capital). If this tax base reacts negatively to an increase in  $\check{\mu}_t^B$ , a standard Laffer curve intuition emerges.

This is indeed the case and easiest to see in steady state. Then  $q^B$  is explicitly given

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<sup>33</sup>In the dynamic model with counter-cyclical idiosyncratic risk, bubble mining may be possible even if this condition is violated because the negative  $\beta$  property relaxes the existence condition of a bubble.

<sup>34</sup>This equation follows immediately from the government budget constraint.

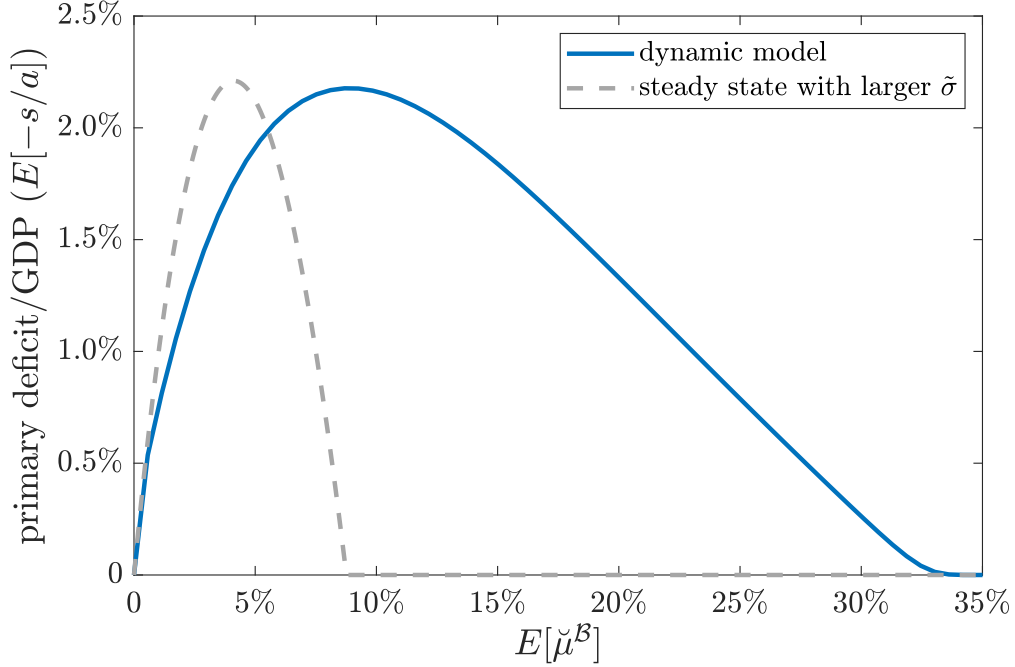


Figure 3: Debt Laffer curve for dynamic model and in steady state when there is a bubble on government debt.  $\tilde{\sigma}$  for the steady state model is increased to match the Laffer curve peak of the dynamic model.

by (for  $\gamma = 1$ )

$$q^B = \frac{(\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \check{\mu}^B})(1 + \phi(a - \mathfrak{g}))}{\sqrt{\rho + \check{\mu}^B} + \phi\rho\bar{\chi}\tilde{\sigma}}.$$

There are two reasons why higher deficits decrease  $q^B$ . First, there is a direct effect from increasing  $\check{\mu}^B$ . This emerges because higher debt growth distorts the portfolio choice between government bonds and capital, making capital more attractive and thereby lowering  $\vartheta$ . If additional deficits are used to lower the output tax rate  $\tau$ , this is the only effect. However, if additional deficits are used to fund government spending by raising  $\mathfrak{g}$ ,  $q^B$  decreases again due to the presence of the term  $a - \mathfrak{g}$  (at least if  $\phi > 0$ ). This second effect is a consequence of the resource constraint (2): when the government claims a larger share of output, consumption has to decline, which lowers all asset values symmetrically.<sup>35</sup>

Outside of the steady state, there is no closed-form solution for  $q^B$  anymore, but the

<sup>35</sup>This intuition breaks down for  $\phi = 0$  as then agents can convert existing capital goods freely into consumption goods and instead the growth rate is reduced.

same Laffer curve logic still applies. The blue line in Figure 3 depicts the debt Laffer curve for the calibrated dynamic model from Section 3. Specifically, this figure plots the average deficit-output ratio that can be sustained for different debt growth policies of the form (12) with identical  $\nu_1$  (identical cyclicalities of debt growth and surpluses) but varying  $\nu_0$ , i.e. the average level of (interest-adjusted) debt growth varies across different policies. The assumption in Figure 3 is that  $g$  remains unchanged, so that larger deficits imply smaller output taxes.

In Figure 3, if the bubble is mined too aggressively so that the average  $\tilde{\mu}^B$  exceeds 8.7%, the government fails to raise additional real revenues. In particular, there is a limit to bubble mining and the government still faces a constraint on real spending. Our calibrated model suggests that the average primary deficit that can be sustained by bubble mining is bounded above by 2.2% of GDP.

It turns out that the negative  $\beta$  property is very important for the qualitative and quantitative shape of the Laffer curve depicted in Figure 3. If we abstracted from counter-cyclical idiosyncratic risk and considered a constant level of  $\tilde{\sigma}$  instead, no permanent deficit could be sustained as the steady-state bubble existence condition  $\tilde{\chi}\tilde{\sigma} > \sqrt{\rho/\gamma}$  is not satisfied for our calibration.

To further understand the importance of the negative  $\beta$  property, we ask by how much we would have to increase the steady-state level of idiosyncratic risk to generate a Laffer curve with the same maximum level of deficits as in the dynamic model. The answer to this question is that idiosyncratic risk would have to be 27% larger. The resulting steady-state Laffer curve is depicted by the gray dashed line in Figure 3. The comparison with the blue line reveals another difference between the dynamic and the steady state model: in the steady-state model the Laffer curve is steeper, so that the tax base is more quickly eroded as the government dilutes the claims of existing bond holders at a faster rate. Instead, in the dynamic model, agents hold on to some bonds even at very large levels of average (interest-adjusted) debt growth rates of more than 10% despite the high inflation rates that they imply. The reason is that the insurance against adverse aggregate events makes bonds attractive for agents even if they pay negative rates of return on average.

## 6 Lessons for Debt Sustainability Analysis

### 6.1 Fiscal Capacity (Off Equilibrium)

Government debt as a bubbly safe asset is only one equilibrium next to possible other no-bubble equilibria. What ensures that the bubble does not burst and that we do not end up with the standard bubble-free real debt valuation, wherein government debt loses its safe-asset status?<sup>36</sup>

First, the government could support the current value of its debt by raising taxes so that it generates a permanently positive surplus stream that backs the current value. This requires that the government has the capacity to raise taxes. Such a policy would of course give up any revenues from bubble mining. Instead it would create a situation where the “conventional” FTPL equation (without second term, the bubble term) applies and determines and supports the price level.

Second, it is sufficient for the government to provide this tax backing *off-equilibrium*. To see this consider the case in which private investors coordinated on the belief that the bubble on government debt was smaller than in the stationary bubble equilibrium and decided to be no longer willing to hold the debt. Then the government could react by permanently reverting to a positive-surplus regime in which debt is fully backed by future surpluses. Such a policy shift would generate capital gains for government debt holders and thus make government debt so attractive *ex ante*, that it would remain optimal for citizens to hold on to their government bonds.

How much *fiscal capacity* is needed to “defend” the bubble on government debt? The off-equilibrium strategy involves permanently positive primary surpluses that grow at the same rate as the economy. While the (positive) scale of these surpluses can be arbitrarily small, the fiscal authority needs the capacity and commitment to turn equilibrium deficits into surpluses before an inflationary collapse of its currency forces it to do so.<sup>37</sup>

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<sup>36</sup>The verbal arguments made in the following two paragraphs are analyzed formally in Brunnermeier et al. (2020) in the context of a steady-state version of our model.

<sup>37</sup>Ultimately, a loss of safe asset status would also force the government to give up bubble mining and reduce the deficit by inflating away the real value of government debt. However, to defend the bubble, the government must revert to surpluses and back the debt at its old, pre-inflation, value to generate capital gains for bond holders that rule out this inflationary equilibrium. The mere ability to raise taxes temporarily when inflation dynamics are underway to stop further inflation is insufficient.

## 6.2 Why is the Safe-asset Bubble not on Private Assets?

So far, our argument does not explain why the safe asset bubble is on government debt and not on any other (private) asset. First, note that the bubble cannot be associated with corporate or household/mortgage debt since they are of finite maturity. While most government bonds are also finite-maturity assets, the difference is that they are a legal claim on money, an infinite-maturity government liability. This leaves only infinitely-lived private equity claims as possible bubble-carrying assets. While theoretically possible, the government could eliminate such a private bubble by following an (off-equilibrium) tax policy that makes its debt a more attractive safe asset than private equity claims. For example, the government could make its off-equilibrium primary surplus stream a much safer asset. Private corporations do not have such an off-equilibrium threat to eliminate all bubbles and therefore cannot force the bubble onto their stocks.<sup>38</sup>

## 7 Conclusion

In this paper we have developed a safe asset theory of government debt based on time-varying idiosyncratic insurance service flows generated by trading government bonds. Our model matches properties of US government debt qualitative and quantitatively and can resolve the empirical puzzles emphasized by Jiang et al. (2019, 2020). The theory also features a novel explanation for the large equity return volatility based on flight to safety into government bonds.

Throughout this paper we have assumed that government bonds are traded on liquid markets. The bubbly safe asset status rests on this assumption because the service flow that citizens derive from government debt is directly tied to their ability to trade it as they experience adverse shocks. The government through its central bank can engage as market maker of last resort so that citizens can trade the asset facing only small bid-ask spreads. This ensures that government debt retains the safe asset status. Private assets do not enjoy this privilege. For example, illiquid long-lived real estate assets

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<sup>38</sup>If a private company ever discovered a technology that generated a sufficiently safe cash flow stream growing at the same rate as the economy, the government could still use countercyclical corporate income taxes to make the company's after-tax cash flows more procyclical and thus the company's stock less suitable as a safe asset.

are unlikely to carry a safe-asset bubble since their large trading costs prevent efficient re-trading.

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## A Appendix

### A.1 Missing Steps in the Derivation of Equation (7)

We present here the missing steps in the derivation of equation (7) left out in the main text. The derivation starts from equation (8), which is a consequence of the intertemporal budget constraint of household  $i$ ,

$$n_0^i = \mathbb{E} \left[ \int_0^\infty \xi_t^i c_t^i dt \right]. \quad (14)$$

To see how this equation implies equation (8), write initial wealth  $n_0^i$  as the sum  $n_0^i = n_0^{b,i} + q_0^K k_0^i$  of initial bond wealth  $n_0^{b,i} := \theta_0^i n_0^i$  and initial capital wealth  $q_0^K k_0^i$ . The value of initial capital wealth must equal the present value of cash flows in and out of the capital portfolio of the agent. Let  $X_t^i$  denote the total cumulative cash flow at time  $t$ , then the cash flow over a small time increment starting at time  $t$  is<sup>39</sup>

$$dX_t^i = \underbrace{a_t k_t^i dt}_{\text{production}} - \underbrace{\iota_t k_t^i dt}_{\text{investment}} - \underbrace{\tau_t a_t k_t^i dt}_{\text{taxes}} - \underbrace{\left( q_t^K k_t^i d\Delta_t^{k,i} + k_t^i d\langle q_t^K, \Delta_t^{k,i} \rangle \right)}_{\text{market purchases of new capital}}.$$

Consequently, initial capital wealth must satisfy<sup>40</sup>

$$\begin{aligned} q_0^K k_0^i &= \mathbb{E} \left[ \int_0^\infty \left( \zeta_t^i dX_t^i + d\langle \zeta_t^i, X_t^i \rangle \right) \right] \\ &= \mathbb{E} \left[ \int_0^\infty \zeta_t^i k_t^i (a - \iota_t - \tau_t a_t) dt \right] \\ &\quad + \mathbb{E} \left[ \int_0^\infty \zeta_t^i k_t^i q_t^K \left( \zeta_t^i \sigma_t^{\Delta,i} + \zeta_t^i \tilde{\sigma}_t^{\Delta,i} - \mu_t^{\Delta,i} - \sigma_t^{\Delta,i} \sigma_t^{q,K} \right) dt \right]. \end{aligned}$$

Here, the last equation uses the notation  $d\Delta_t^{k,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t$ . Substituting the previous equation into the the intertemporal budget constraint (14) yields

$$n_0^{b,i} = \mathbb{E} \left[ \int_0^\infty \zeta_t^i \left( c_t^i - k_t^i (a - \iota_t - \tau_t a_t) + q_t^K k_t^i \left( \mu_t^{\Delta,i} + \sigma_t^{\Delta,i} \sigma_t^{q,K} - \zeta_t^i \sigma_t^{\Delta,i} - \zeta_t^i \tilde{\sigma}_t^{\Delta,i} \right) \right) dt \right],$$

which implies equation (8) if we can show that

$$\mu_t^{\Delta,i} + \sigma_t^{\Delta,i} \sigma_t^{q,K} - \zeta_t^i \sigma_t^{\Delta,i} - \zeta_t^i \tilde{\sigma}_t^{\Delta,i} = (1 - \theta_t) \bar{\chi} \tilde{\sigma}_t \tilde{\sigma}_t^{\Delta,i}. \quad (15)$$

<sup>39</sup>Intuitively, the last quadratic covariation term results from the fact that market transactions are made “at the end of the period”. It can be easiest seen when writing the cash flows from market purchases in pseudo discrete time notation:

$$q_{t+dt}^K k_t^i (\Delta_{t+dt}^i - \Delta_t^i) = q_t^K k_t (\Delta_{t+dt}^i - \Delta_t^i) + k_t^i (q_{t+dt}^K - q_t^K) (\Delta_{t+dt}^i - \Delta_t^i).$$

<sup>40</sup>This is the correct present value formula for non-absolutely continuous cumulative cash flows. Again, the less familiar quadratic covariation term appearing here can easiest be understood when considering the pseudo discrete time version:

$$PV(dX) = \sum_t \zeta_{t+dt} (X_{t+dt} - X_t) = \sum_t \left( \zeta_t (X_{t+dt} - X_t) + (\zeta_{t+dt} - \zeta_t) (X_{t+dt} - X_t) \right)$$

TO ADD: proof of (15)!

ANOTHER THING TO ADD (because we claim it in the main text): proof of  $\tilde{\sigma}_t^{\Delta,i} = -\vartheta_t \tilde{\chi} \tilde{\sigma}_t$

To simplify the integrand of the second term, we need to determine  $\mu_t^\Delta$ ,  $\sigma_t^\Delta$ , and  $\tilde{\sigma}_t^\Delta$ . In equilibrium, we have  $q_t^K k_t^i = (1 - \vartheta_t) n_t^i$  and applying Ito to both sides of this equation yields

$$\begin{aligned} d(q_t^K k_t^i) &= q_t^K dk_t^i + k_t^i dq_t^K + d\langle q_t^K, k_t^i \rangle \\ &= q_t^K k_t^i \left( \left( \Phi(l_t) - \delta + \mu_t^{q,K} + \mu_t^{\Delta,i} + \sigma_t^{q,K} \sigma_t^{\Delta,i} \right) dt + \left( \sigma_t^{q,K} + \sigma_t^{\Delta,i} \right) dZ_t + \left( \tilde{\sigma}_t + \tilde{\sigma}_t^{\Delta,i} \right) d\tilde{Z}_t^i \right) \end{aligned}$$

$$\begin{aligned} d\left( (1 - \vartheta_t) n_t^i \right) &= (1 - \vartheta_t) dn_t^i - n_t^i d\vartheta_t - d\langle \vartheta_t, n_t^i \rangle \\ &= q_t^K k_t^i \left( \left( -\rho + r_t^n - \frac{\vartheta_t \mu_t^\vartheta}{1 - \vartheta_t} - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} \sigma_t^n \right) dt + \left( \sigma_t^n - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} \right) dZ_t + \tilde{\sigma}_t^n d\tilde{Z}_t^i \right) \end{aligned}$$

Comparing terms yields

$$\tilde{\sigma}_t^{\Delta,i} = \tilde{\sigma}_t^n - \tilde{\sigma}_t = (1 - \vartheta_t) \tilde{\sigma}_t - \tilde{\sigma}_t = -\vartheta_t \tilde{\sigma}_t,$$

$$\sigma_t^{\Delta,i} = \sigma_t^n - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} - \sigma_t^{q,K} = \sigma_t^n - \vartheta_t \sigma^{q,B} - (1 - \vartheta_t) \sigma_t^{q,K} = 0,$$

and

$$\mu_t^{\Delta,i} = -\rho + r_t^n - \frac{\vartheta_t \mu_t^\vartheta}{1 - \vartheta_t} - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} \sigma_t^n - \left( \Phi(l_t) - \delta + \mu_t^{q,K} \right) = 0$$

(these require using the definition of  $\vartheta$  plus computing the average return on wealth  $r_t^n$ ). Intuitively, all this makes of course a lot of sense: because all households are the same up to scale, they do not trade because of expected time trends ( $\mu^{\Delta,i} = 0$ ) or in response to aggregate shocks ( $\sigma^{\Delta,i} = 0$ ), but only in response to idiosyncratic shocks. When they receive a positive shock, they sell capital (negative  $\tilde{\sigma}_t^{\Delta,i}$ ), when they receive a negative shock, they buy capital (positive  $\tilde{\sigma}_t^{\Delta,i}$ ). The magnitude of trading  $\tilde{\sigma}_t^{\Delta,i}$  in response to idiosyncratic shocks is proportional to the portfolio weight on bonds  $\vartheta_t$  in households' portfolios.

TODO: streamline this! And add the  $\bar{\chi}$  everywhere...

## A.2 Model Solution with Stochastic Differential Utility

TO BE ADDED HERE: the model solution with EZ utility. Most of the steps are the same as in the main text, just have a slightly more complicated HJB, the functional form guess is  $V_t(n^i) = \frac{(\omega_t n^i)^{1-\gamma}}{1-\gamma}$  and the risk premium terms have an additional  $\sigma_t^\omega$  in some places...