

# **Initial results examining the Equity Premium puzzle using Habit Formation and non-Gaussian stock price movement**

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## **Abstract**

Mehra and Prescott (1985) found that for reasonable values of a relative risk aversion coefficient (=RRA) and given small variance of the growth rate in per capita consumption, an Equity Premium puzzle exists. In other words, the difference between the expected rate of return on the stock market and the riskless rate of interest ( $= \mu - r$ ) is too big. I will use a model with no time separability of preferences and with habit persistence (= adjacent complementarity in consumption) to try and explain the equity premium puzzle. In addition, and unique to this paper, I represent the stock price movement using a right skewed non-Gaussian model (the Gumbel distribution). My results suggest that habit persistence can generate the Mehra and Prescott's (1985) sample mean and variance of the consumption growth rate with low risk aversion, hence giving a plausible explanation to the equity premium puzzle.

## **I) Introduction**

The equity premium puzzle emanates from the inability of the theoretical models to explain the empirically observed high equity premium (when the average stock returns so much higher than the average bond returns). It is based on the fact that in order to reconcile the much higher return on stock compared to government bonds in the United States, individuals must have very high risk aversion according to standard economics models.

Mehra and Prescott (1985) reported the existence of an equity premium puzzle. They calibrated model parameters to match the sample mean, variance, and first order autocorrelation of the annual growth rate of per capita consumption in the years 1889-1978. They used a time- and state-separable utility. They were unable to determine pairs of discount rate and relative risk aversion (RRA) coefficients to match the sample mean of the annual real interest rate and of the equity premium between 1889-1978, hence a puzzle. In other words, the consumption growth rate was too smooth to justify the average equity premium.

One type of tests, to resolve the equity premium puzzle, examines the Euler equation restriction on the product of asset returns with the marginal rate of substitution of the representative agent. Habit persistence usually causes specific lag coefficients in the Euler equation to be negative. Positive coefficients in the Euler equation are considered to be evidence of durability of goods, rather than habit persistence. Ferson and Constantinides (1989) used quarterly and annual data

and found negative coefficients in the Euler equation which supports habit persistence. They also rejected the time separable model and supported instead a model with habit persistence with respect of explaining the equity premium puzzle (as in this paper).

Hansen and Jagannathan (1988), using monthly data, support habit persistence. Heaton (1988) found evidence to support habit persistence by checking the autocorrelations in consumption, while considering time aggregation.

However other papers, such as Dunn and Singleton (1986), Gallant and Tauchen (1989), Eichenbaum Hansen and Singleton (1988) and Eichen and Hansen (1990), found that specific lag coefficients in the Euler equation to be positive which support durability of goods rather than habit persistence.

In addition, Hansen and Singleton (1982, 1983), Ferson (1983), Grossman, Melino, and Shiller (1987) rejected the Euler equation restriction on asset returns and the marginal rate of substitution caused by the time- and state-separable preferences.

Shiller (1981) found that stock market returns are too volatile relative to the volatility of dividends.

Duesenberry (1949) is the first comprehensive examination of the effects of habit persistence. Duesenberry's demonstration effect is a type of consumption externality where an individual's utility depends not only upon his consumption level ( $C_t$ ) but also upon the social average level (or Habit formation) of consumption ( $\bar{C}_t$ ). Duesenberry (1949) claims that the demonstration effect causes unhappiness with current levels of consumption, which affects savings rates and macroeconomic growth.

The Brock (1982) asset pricing model can estimate a significant equity premium. Using the Brock's (1982) asset pricing model, Akdeniz and Dechert (2007) show that there are parameterizations of the Brock model that have equity premia that are more consistent with the empirical evidence than the equity premia that were observed by Mehra and Prescott (1985). Kocherlakota (1996) tries to resolve the equity premium puzzle and the risk free rate puzzle by reviewing the literature. Kocherlakota report the papers that try to explain these two puzzles. Campbell and Cochrane (1999) find the equity premium puzzle using a consumption-based model with external habit formation. They use a power utility and the Sharpe ratio inequality to explain the equity premium puzzle.

In this paper, I represent the stock price movement using a right skewed non-Gaussian model (the Gumbel distribution) unlike other papers, such as Constantinides (1990), that use a Gaussian model for this purpose. To better represent a risk averse investor, I use a right skew distribution for the stock price since such a distribution gives higher weight to negative stock return than positive stock return.

The objective of this paper is to explain the equity premium puzzle in a rational expectations model with time nonseparability of preferences and with habit persistence (i.e. adjacent complementarity in consumption) while representing the stock price movement using a right skewed non-Gaussian model (the Gumbel distribution).

The paper is organized as follows: In section II, the theoretical model and its assumptions are presented. In section III, I find an optimal Consumption policy and an optimal Investment policy. In section IV, I find the relation between the RRA coefficient and the intertemporal Elasticity of Substitution in Consumption. In section V, I attempt to explain the Equity Premium Puzzle. In section VI, I examine the effect of time separability in utility preferences on the Equity Premium puzzle. In section VII, I examine the effect of both time separability in utility preferences and Habit persistence on the Equity Premium puzzle. In section VIII, I present my conclusions.

## II) The Model and Assumptions

I assume the utility function for the representative agent in this economy to be of the following power type:

$$(1) \quad u(c(t), x(t)) = \frac{1}{\gamma} [c(t) - x(t)]^\gamma$$

where  $x(t)$  is the subsistence level of per capita consumption generated by Habit persistence (i.e. the habit-forming state variable),  $c(t)$  is the level of per capita consumption,  $\gamma$  is the utility sensitivity to the difference between consumption and the subsistence level of consumption.

$x(t)$  is defined by:

$$(2) \quad x(t) = x_0 e^{-at} + b \int_0^t e^{a(s-t)} c(s) ds$$

The special case of  $b = 0$  corresponds to a time-separable utility.

Thus, for a time-separable utility, the coefficient of absolute risk aversion is:

$$A[c(t) - x(t)] = - \left[ \frac{u''(c_t, x_t)}{u'(c_t, x_t)} \right] = - \left[ \frac{(\gamma - 1) [c(t) - x(t)]^{\gamma-2}}{[c(t) - x(t)]^{\gamma-1}} \right] = \frac{1 - \gamma}{c(t) - x(t)}$$

The coefficient of relative risk aversion is:

$$R[c(t) - x(t)] = RRA \text{ with respect to } c - x = [c(t) - x(t)] A(c_t) = 1 - \gamma$$

Another RRA definition :

$$\begin{aligned} RRA \text{ with respect to } x &= x(t) \left[ - \frac{u_{xx}}{u_x} \right] = x(t) \left[ \frac{-(\gamma - 1) [c(t) - x(t)]^{\gamma-2}}{(-1)[c(t) - x(t)]^{\gamma-1}} \right] = \\ &= (\gamma - 1) \frac{x(t)}{c(t) - x(t)} = (\gamma - 1)y(t) \end{aligned}$$

For simplicity, I assume that there are only two assets (without loss of generality), one risky stock and one risk-free bond, to invest in.

To better represent a risk averse investor, I use a right skew distribution for the stock price since such a distribution gives higher weight to negative stock return than positive stock return.

To represent the stock price motion I use the Gumble probability distribution which is a non-Gaussian distribution with an asymmetric right skew. Thus, the formulation of the stock price  $S_t$  is:

$$(3) \quad dS_t = [\mu_t + \varepsilon\sigma_t]S_t dt + \frac{\pi}{\sqrt{6}}\sigma_t S_t dw_t$$

Where  $\frac{\pi}{\sqrt{6}}\sigma_t$  is the Gumble distribution standard deviation,  $\mu_t + \varepsilon\sigma_t$  is the Gumble distribution mean,  $\varepsilon = 0.577216$  the Euler's constant,  $\sigma_t$  is the distribution scale (i.e. the normal distribution standard deviation) of the price of stock (which is a risky asset),  $\mu_t$  is the distribution location (i.e. the normal distribution mean) of the risky rate of return and  $w_t$  is a Wiener process.

The percent change, i.e. the return, of the stock price is described as:

$$(4) \quad \frac{dS_t}{S_t} = [\mu + \varepsilon\sigma]dt + \frac{\pi}{\sqrt{6}}\sigma dw_t$$

The infinitely lived representative consumer has wealth  $W_t$ .

The change in the per capita wealth ( $W_t$ ) is determined by the return from the risky asset (the stock) and the return from the riskless asset (the risk-free bond) less consumption. Hence:

$$(5) \quad dW_t = (1 - \alpha_t)W_t r dt + \alpha_t W_t \frac{dS_t}{S_t} - c_t dt$$

where  $r$  is risk-free rate of return,  $C_t$  is per capita consumption, and  $\alpha_t$  is the portion, i.e. weight, of the per capita wealth ( $W_t$ ) that is invested in the risky asset.

Inserting equation (4) into equation (5) yields the equation governing the changes in the wealth written as follows:

$$(6) \quad dW_t = \{[\alpha_t(\mu - r + \varepsilon\sigma) + r]W_t - c_t\}dt + \left(\frac{\pi}{\sqrt{6}}\right)\alpha_t W_t \sigma dw_t$$

where  $W_t$  is the per capita wealth,  $r$  is risk-free rate of return,  $\mu$  is risky rate of return,  $\alpha_t$  is the proportion of wealth invested in the risky asset, and  $w_t$  is a Wiener process.

My goal is not to study the most general utility function that exhibits habit persistence but rather to use the most simple utility specification that may explain the equity premium puzzle.

If  $\alpha_t = 0$ , equation (6) becomes:

$$(7) \quad dW_t = \{rW_t - c_t\}dt$$

Thus 
$$\frac{dW_t}{dt} = rW_t - c_t$$

Using a constant wealth path in equation (7), i.e.  $\frac{dW_t}{dt} = 0$ , I derive

$$c_t = rW_t$$

For a particular consumption policy (when  $\alpha_t = 0$ ) where

$$W_t = W_0 e^{(b-a)t} > 0$$

and

$$c_t = (r + a - b)W_t = (r + a - b)W_0 e^{(b-a)t} > 0$$

thus, using equation (2) I derive

(8)

$$\begin{aligned} c(t) - x(t) &= (r + a - b)W_0 e^{(b-a)t} - \left[ x_0 e^{-at} + b \int_0^t e^{a(s-t)} c(s) ds \right] \\ &= (r + a - b) \left[ W(t) - \frac{x(t)}{r + a - b} \right] \end{aligned}$$

For a time-separable utility (i.e.  $b = 0$ ) I derive

$$W_t = W_0 e^{(b-a)t} = W_0 e^{(-a)t}$$

$$x(t) = x_0 e^{-at} + b \int_0^t e^{a(s-t)} c(s) ds = x_0 e^{-at}$$

Hence

$$c(t) - x(t) = (r + a - b)W_0 e^{(-a)t} - x_0 e^{-at} = (r + a - b)e^{(-a)t} \left[ W_0 - \frac{x_0}{r+a-b} \right]$$

To require  $c(t) - x(t) > 0$  I need to make the following conditions:

$$(9) \quad W_0 > 0$$

$$(10) \quad W_0 - \frac{x_0}{r+a-b} > 0$$

$$(11) \quad r + a - b > 0$$

Or

$$r + a > b > 0$$

In addition, I can use a constant consumption path which can be described as

$$(12) \quad c(t) = \frac{ax(t)}{b}$$

or

$$bc(t) - ax(t) = 0$$

### III) Finding an optimal Consumption policy ( $c_t^*$ ) and Investment policy ( $\alpha_t^*$ )

**Theorem 1** shows existence and uniqueness of an optimal policy which determine the utility of capital, and the dynamics of capital and consumption.

From the proof of **Theorem 1 in Appendix A**, I find the following optimal Consumption policy ( $c_t^*$ ) and Investment policy ( $\alpha_t^*$ ):

$$(13) \text{ (A17)} \quad c^*(t) = x(t) - (h)^{\frac{\gamma}{\gamma-1}} \left[ W(t) - \frac{x(t)}{r+a-b} \right] \left[ \frac{(r+a)}{H(r+a-b)} \right]$$

$$(14) \text{ (A5)} \quad \alpha^*(t) = m \left[ 1 - \frac{\frac{x(t)}{W(t)}}{r+a-b} \right]$$

$$(15) \text{ (A19)} \quad m = \frac{(\mu-r+\varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \quad \text{where } 0 \leq m \leq 1$$

$$(16) \text{ (A20)} \quad h = \frac{r+a-b}{(a+r)(1-\gamma)} \left\{ \rho - \gamma r - \frac{(\mu-r+\varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\} > 0$$

The utility of wealth (=capital):

$$(17) \text{ (A10)} \quad V[W(t), x(t)] = \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma \left( \frac{-1}{H} \right)$$

$$(18) \text{ (A21)} \quad H = \left( \frac{1}{1-\gamma} \right) \left\{ \rho - \gamma r - \frac{9\gamma(\mu-r+\varepsilon\sigma)^2}{2(\pi^2)\sigma^2(1-\gamma)} \right\}$$

The wealth (=capital) is

$$(19) \text{ (A24)} \quad W(t) = \frac{x(t)}{r+a-b} + \left[ W(0) - \frac{x(0)}{r+a-b} \right] e^{\left[ nt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m w(t) \right]}$$

where

$$(20) \text{ (A22)} \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

The above optimal policies  $(c^*, \alpha^*)$  are unique as I show in appendix A.

The consumption growth rate is

$$(21) \text{ (A30)} \quad \frac{dc(t)}{c(t)} = \left[ n + b - \frac{(n+a)x(t)}{c(t)} \right] dt + \left[ 1 - \frac{x(t)}{c(t)} \right] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

I use the optimal policies found in theorem 1 as the equilibrium paths in a representative-consumer economy. Specifically the consumption growth rate (defined in equation 21) is considered to be the per capita consumption growth rate.

I will now define  $z(t)$  the subsistence rate of consumption generated by Habit Persistence.

$$(22) \quad z(t) = \frac{x(t)}{c(t)}$$

Where  $x(t)$  is the subsistence level of consumption generated by Habit Persistence.

Since

$$(23) \quad \text{RRA with respect to } x = (\gamma - 1) \frac{x(t)}{c(t) - x(t)}$$

I will define

$$(24) \quad y(t) = \frac{x(t)}{c(t) - x(t)} = \frac{z(t)}{1 - z(t)}$$

**Theorem 2** derives the stationary distribution of the state variable ( $= z$ ) and calculates the unconditional mean and variance of the consumption growth rate.

From **Theorem 2 proof in Appendix B**,  $y(t)$  has a stationary probability distribution with density

$$(25) \text{ (B14)} \quad p_y(y_t, t) = ky \left( \frac{-12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right) e^{\left\{ \frac{-12b}{\pi^2 \sigma^2 m^2 y} \right\}}$$

where

$$(26) \text{ (B15)} \quad k^{-1} = \left( \frac{12b}{\pi^2 \sigma^2 m^2} \right) \left\{ 1 - \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} \Gamma \left\{ \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} - 1 \right\}$$

And  $\Gamma(\cdot)$  is the gamma function.

Using the above probability distribution density I will calculate the following:  
 $y(t)$  has a single mode (see appendix B)

$$(27) \text{ (B12)} \quad \hat{y} = \frac{b}{n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2} < \infty$$

Thus requires the condition  $n + a - b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 > 0$

And a mean

$$(28) \text{ (B11)} \quad \bar{y} = \frac{b}{n+a-b} < \infty$$

Thus requires the condition  $n + a - b > 0$

$z(t) = \frac{x(t)}{c(t)}$  has a stationary probability distribution with the density in equation (29)

$$(29) \text{ (B17)} \quad p_z(z_t, t) = k e^{\left\{ \frac{12b}{\pi^2 \sigma^2 m^2} \right\}} (1-z) \left[ \frac{12(b+a-b)}{\pi^2 \sigma^2 m^2} \right] z \left\{ \frac{12 \left[ b-a-n - \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} e^{\left\{ \frac{-12b}{\pi^2 \sigma^2 m^2 z} \right\}}$$

Where  $0 \leq z < 1$

For the stationary distribution,  $z(t)$  has a single mode

$$(30) \text{ (B19)} \quad \hat{z} = \frac{\left[ n+a + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right] - \sqrt{\left[ n+a + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]^2 - 4 \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 b}}{2 \left( \frac{\pi^2}{6} \right) \sigma^2 m^2}$$

Next, I will derive the unconditional mean of consumption growth  $\frac{E \left[ \frac{dc(t)}{c(t)} \right]}{dt}$   
**In appendix C**, i derive

$$(31) \text{ (C1)} \quad \frac{E \left[ \frac{dc(t)}{c(t)} \right]}{dt} = n + b - (n+a) E[z(t)] = n + b - (n+a) \int_0^1 z(t) p_z(z_t, t) dz$$



Since a closed form expression for the integral is unavailable, the integration is done numerically.

Next, I will derive the unconditional variance of consumption growth  $\frac{var[\frac{dc(t)}{c(t)}]}{dt}$   
 In appendix C, i derive

(32) (C2)

$$\frac{var[\frac{dc(t)}{c(t)}]}{dt} = \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 E\{[1 - z(t)]^2\} = \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 \int_0^1 [1 - z(t)]^2 p_z(z_t, t) dz$$

Since a closed form expression for the integral is unavailable, the integration is done numerically.

#### IV) The relation between the RRA coefficient and the intertemporal Elasticity of Substitution in Consumption (=s)

Constantinides (1990) asserts that “Habit Persistence drives a wedge between the RRA coefficient and the inverse of the intertemporal Elasticity of Substitution in Consumption (=s)”. That means that the relation can be described as

$$(32.5) \quad RRA = wedge * \frac{1}{s}$$

Or

$$RRA * s = wedge$$

For a time-separable model (b=0),  $RRA * s = 1$ . I will show that for a nonseparable model and for specific variable values that may explain the equity premium puzzle the wedge is extremely below one i.e.  $RRA * s \ll 1$ .

In **appendix D**, I derive the expressions for RRA coefficient and s (=intertemporal Elasticity of Substitution in Consumption).

RRA coefficient defined as

(33) (D1)

$$RRA = \frac{-WV_{WW}}{V_W} = (1 - \gamma) \frac{1}{1 - \frac{x(t)}{W(r + a - b)}}$$

In the short run, a shock of a decrease in wealth does not change  $x(t)$  but increases the RRA coefficient. In the long run, the stationary distribution of the RRA coefficient decreases the RRA coefficient back.

thus

$$(34) \text{ (D3)} \quad RRA = (1 - \gamma) \left\{ 1 + y(t) \left[ \frac{h}{(r+a-b)} \right] \right\}$$

Since  $y(t)$  has a steady state distribution, so does the RRA coefficients. Thus,

$$(35) \text{ (D4)} \quad \overline{RRA} = (1 - \gamma) \left\{ 1 + \left[ \frac{hb}{(n+a-b)(r+a-b)} \right] \right\}$$

When  $z(t) = \hat{z}$  (=modal value of  $z$ )

$$(36) \text{ (D5)} \quad RRA(z = \hat{z}) = (1 - \gamma) \left\{ 1 + \left[ \frac{\hat{z}}{1-\hat{z}} \right] \left[ \frac{h}{(r+a-b)} \right] \right\}$$

$$(37) \text{ (D6)} \quad s = \frac{\partial \left[ \frac{E(\frac{dc}{c})}{dt} \right]}{\partial r} = \frac{1-z(t)}{1-\gamma} \quad \text{when } z(t), \mu - r, \text{ and } \sigma^2 \text{ held constant}$$

$$(38) \text{ (D7)} \quad s * RRA = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} z(t)$$

The modal value of  $s*RRA$  is

$$(39) \text{ (D8)} \quad mode(s * RRA) = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} \hat{z}$$

For the equilibrium of my specific model I can write the following equations

$$(40) \quad s * RRA = \frac{\partial c/c}{\partial W/W} \text{ when } x(t) \text{ is held constant}$$

$$(41) \quad s * RRA = \frac{std(\partial c/c)}{std(\partial W/W)}$$

Which is the ratio of the standard deviation of the consumption growth rate and the standard deviation of the capital (wealth) growth.

## V) Examining the Equity Premium Puzzle

Mehra and Prescott (1985) estimated the mean of the annual growth rate of per capita real consumption of nondurables and services in the years 1889-1978 to be 0.0183 per year. Thus I will use his estimate and set

$$(42) \quad \frac{E\left[\frac{dc(t)}{c(t)}\right]}{dt} = 0.0183 \text{ per year.}$$

In addition, they also estimated the standard deviation of the growth rate in the years 1889-1978 to be 0.0357 per year. Thus I will use his estimate and set

$$(43) \quad \frac{\text{var}\left[\frac{dc(t)}{c(t)}\right]}{dt} = (0.0357)^2 \text{ per year.}$$

Mehra and Prescott (1985) also estimated the mean annual real rate of return on a relatively riskless security to be 0.008. Thus I will set

$$(44) \quad r = 0.01 \text{ per year.}$$

This paper uses an economy that allows for production, thus I define  $K(t)$  as the capital of a firm. The firm invest  $\delta_1 K(t)$  capital in risky technology and  $(1 - \delta_1)K(t)$  capital in riskless technology. Thus  $0 < \delta_1 \leq 1$ . The firm can finance its activity with equity ( $= S(t)$ ) and with riskless debt ( $= B(t)$ ). The firm ratio of equity value to entire capital is  $\frac{S(t)}{S(t)+B(t)} = \delta_2$  which is kept constant and it's domain is  $0 < \delta_2 \leq 1$ . The firm's capital ( $= K(t)$ ) is equal to the sum of the value of equity and the value of riskless debt, i.e.  $[K(t) = S(t) + B(t)]$ . Since bonds are riskless, the rate of return for riskless assets is estimated by the bonds return which is equal to  $dB(t)/B(t) = rdt$ . Since equity is risky, the rate of return for risky assets is estimated by the equity return which is equal to  $dS(t)/S(t)$ .

The total capital change of the firm should equal to the returns from its investments thus

$$(45) \quad dS(t) + B(t)rdt = \delta_1 K(t) \frac{dS(t)}{S(t)} + (1 - \delta_1)K(t)rdt$$

From equation (4)

$$(4) \quad \frac{dS(t)}{S(t)} = (\mu + \varepsilon\sigma)dt + \frac{\pi}{\sqrt{6}}\sigma dw(t)$$

Insert equation (4) into (45)

$$(45) \quad dS(t) + B(t)rdt = \delta_1 K(t) \left[ (\mu + \varepsilon\sigma)dt + \frac{\pi}{\sqrt{6}}\sigma dw(t) \right] + (1 - \delta_1)K(t)rdt$$

Divide both sides by  $S(t)$  to derive

$$(46) \quad \frac{dS(t)}{S(t)} + \frac{B(t)}{S(t)}rdt = \delta_1 \frac{K(t)}{S(t)} \left[ (\mu + \varepsilon\sigma)dt + \frac{\pi}{\sqrt{6}}\sigma dw(t) \right] + (1 - \delta_1) \frac{K(t)}{S(t)}rdt$$

$$\frac{dS(t)}{S(t)} + \left[ \frac{1}{\delta_2} - 1 \right] r dt = \delta_1 \frac{1}{\delta_2} \left[ (\mu + \varepsilon\sigma) dt + \frac{\pi}{\sqrt{6}} \sigma dw(t) \right] + (1 - \delta_1) \frac{1}{\delta_2} r dt$$

thus

$$(47) \quad \frac{dS(t)}{S(t)} = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right] dt + \left( \frac{\delta_1}{\delta_2} \right) \left( \frac{\pi}{\sqrt{6}} \right) \sigma dw(t)$$

In **appendix E** I derive

$$(48) \text{ (E1)} \quad \frac{E\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right]$$

$$(49) \text{ (E2)} \quad \frac{\text{var}\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left( \frac{\delta_1}{\delta_2} \right)^2 \left( \frac{\pi^2}{6} \right) \sigma^2$$

Mehra and Prescott (1985) estimated the annual real return on the Standard and Poor's composite stock price index in the 1889-1978 period to have a mean 0.0698 and a standard deviation 0.1654. Ibbotson and Sinquefeld (1982) had generally similar estimates. Thus I set

$$(50) \quad \frac{E\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right] = 0.06 \text{ per year}$$

Inserting equation (44) I derive

$$(51) \quad \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) = 0.06 - r = 0.06 - 0.01 = 0.05$$

$$(52) \quad \frac{\text{var}\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left( \frac{\delta_1}{\delta_2} \right)^2 \left( \frac{\pi^2}{6} \right) \sigma^2 = (0.165)^2 \text{ per year}$$

From equation (15)

$$(15) \text{ (A19)} \quad m = \frac{(\mu - r + \varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \quad \text{where } 0 \leq m \leq 1$$

I can set the condition

$$(53) \quad m = \frac{(\mu - r + \varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} < 1$$

Using equations (51) and (52) I derive

$$(1 - \gamma) \geq \frac{(\mu - r + \varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2} = \frac{\frac{\delta_1}{\delta_2} \left[ \frac{\delta_1}{\delta_2} (\mu - r + \varepsilon\sigma) \right]}{\left(\frac{\delta_1}{\delta_2}\right)^2 \left(\frac{\pi^2}{6}\right)\sigma^2} = \frac{\left(\frac{\delta_1}{\delta_2}\right) 0.05}{(0.165)^2} = \left(\frac{\delta_1}{\delta_2}\right) 1.836$$

Thus

$$(54) \quad (1 - \gamma) \geq \left(\frac{\delta_1}{\delta_2}\right) 1.836$$

Using equation (32) and (15) I drive

$$(55) \quad \frac{\text{var}\left[\frac{dc(t)}{c(t)}\right]}{dt} = \left(\frac{\pi^2}{6}\right)\sigma^2 m^2 = \left(\frac{\pi^2}{6}\right)\sigma^2 \frac{(\mu - r + \varepsilon\sigma)^2}{\left(\frac{\pi^2}{36}\right)\sigma^4 (1 - \gamma)^2} = \frac{6(\mu - r + \varepsilon\sigma)^2}{\pi^2 \sigma^2 (1 - \gamma)^2}$$

From equation (43)

$$(43) \quad \frac{\text{var}\left[\frac{dc(t)}{c(t)}\right]}{dt} = (0.0357)^2$$

insert equations (43) into (55)

$$(56) \quad \frac{\text{var}\left[\frac{dc(t)}{c(t)}\right]}{dt} = \frac{6(\mu - r + \varepsilon\sigma)^2}{\pi^2 \sigma^2 (1 - \gamma)^2} = (0.0357)^2$$

Multiply numerator and denominator by  $\left(\frac{\delta_1}{\delta_2}\right)^2$

$$\frac{\left(\frac{\delta_1}{\delta_2}\right)^2 (\mu - r + \varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right) \left(\frac{\delta_1}{\delta_2}\right)^2 \sigma^2 (1 - \gamma)^2} = (0.0357)^2$$

Insert equations (51) and (52) into the above expression

$$\frac{(0.05)^2}{(0.165)^2 (1 - \gamma)^2} = (0.0357)^2$$

Solving for  $1 - \gamma$

$$(57) \quad 1 - \gamma = 8.488$$

I now intend to show that habit persistence can generate the Mahra and Prescott's sample mean and variance of the consumption growth rate with a lower RRA coefficient i.e. with a lower  $(1 - \gamma)$  see equation (58).

To determine how low a  $(1 - \gamma)$  I can choose I do the following.

Using equation (34) and (57)

$$(58) \quad RRA = (1 - \gamma) \left\{ 1 + y(t) \left[ \frac{h}{(r+a-b)} \right] \right\} > (1 - \gamma) = 8.488$$

Since  $y(t) > 0$ ,  $h > 0$ , and  $r + a - b > 0$  according to conditions above.

From equation (54)

$$(54) \quad (1 - \gamma) \geq \left( \frac{\delta_1}{\delta_2} \right) 1.836$$

Inserting equation (57) into (58)

$$8.488 \geq \left( \frac{\delta_1}{\delta_2} \right) 1.836$$

Thus

$$(59) \quad 4.6233 \geq \left( \frac{\delta_1}{\delta_2} \right)$$

I thus set  $\frac{\delta_1}{\delta_2} = 1$  which is consistent with equation (59) and will determine a lower value of  $(1 - \gamma)$  that I intend to use. Thus from equation (54) I derive

$$(60) \quad (1 - \gamma) \geq \left( \frac{\delta_1}{\delta_2} \right) 1.836 = 1.836$$

For simplicity I choose

$$(61) \quad (1 - \gamma) = 1.836$$

Hence from equation (58) I can derive the following regarding the specific lower  $(1 - \gamma)$  I use.

$$(62) \quad RRA > (1 - \gamma) = 1.836$$

I set the rate of time preference in units  $(year)^{-1}$  [i.e. the constant utility discount rate see equation (A8)] to

$$(63) \quad \rho = 0.037.$$

I proceed with creating Table 1 using the equations found above and the following settings.

The annual rate of return of the riskless technology=  $r=0.01$ , the power in the utility function=  $= -0.836$  , the rate of time preference in units  $(year)^{-1}=\rho = 0.037$  ,

the unconditional mean of the annual growth rate in consumption (line 4 table 1) =  $\frac{E\left[\frac{dc(t)}{c(t)}\right]}{dt} = 0.018$  ,

the unconditional standard deviation of the annual growth rate in consumption (line 6 table 1) =  $\frac{var\left[\frac{dc(t)}{c(t)}\right]}{dt} = (0.036)^2$  , and equations (51) and (52).

I have chosen pairs of parameters (a, b) that meet the conditions in equations (11) and (28) and for which the mean and variance of the consumption growth rate [equations (31) and (32)] match their sample estimates [equations (42) and (43)].

Table1							
Mean and Variance of the Consumption Growth Rate generated by the model with Habit resistance							
Parameter a, per year		0.1	0.2	0.3	0.4	0.5	0.6
Parameter b		0.093	0.172	0.25	0.328	0.405	0.492
Mode(z)		0.68	0.73	0.75	0.75	0.76	0.78
Mean annual growth rate in consumption:							
	unconditional mean	0.018	0.019	0.018	0.018	0.018	0.018
	at z= mode(z)	0.034	0.034	0.034	0.034	0.034	0.033
Standard deviation of the annual growth rate in							
	unconditional mean	0.036	0.036	0.036	0.036	0.036	0.034
	at z= mode(z)	0.053	0.045	0.042	0.041	0.040	0.037
RRA coefficient:							
	unconditional mean	3.21	2.67	2.43	2.30	2.21	2.18
	at z= mode(z)	2.94	2.58	2.38	2.27	2.19	2.16
Elasticity of substitution (s):							
	at z= mode(z)	0.18	0.15	0.14	0.13	0.13	0.12
s*RRA at z=mode(z):		0.52	0.38	0.33	0.30	0.29	0.26

The modal value of  $z(t)$  (line 3 table 1) is calculated using equation (30)

$$(30) \text{ (B19)} \quad \hat{z} = \frac{\left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right] - \sqrt{\left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right]^2 - 4\left(\frac{\pi^2}{6}\right)\sigma^2m^2b}}{2\left(\frac{\pi^2}{6}\right)\sigma^2m^2}$$

Where

$$(15) \text{ (A19)} \quad m = \frac{(\mu-r+\varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)}$$

$$(20) (A22) \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

The mean of the annual growth rate in consumption at  $z(t) = \hat{z}$  (line 5 table 1) =  $\frac{E\left[\frac{dc(t)}{c(t)}\right]}{dt}$  at  $\hat{z} = n + b - (n + a)\hat{z}$  [equation (30) (C1)]

The standard deviation of the annual growth rate in consumption at  $z(t) = \hat{z}$  (line 7 table 1) =  $\frac{var\left[\frac{dc(t)}{c(t)}\right]}{dt}$  at  $\hat{z} = \left(\frac{\pi^2}{6}\right)\sigma^2 m^2 [1 - \hat{z}]^2$  [equation (32) (C2)]

The unconditional mean of the RRA coefficient (line 8 table 1) is calculated using equation (35)

$$(35) (D4) \quad \overline{RRA} = (1 - \gamma) \left\{ 1 + \left[ \frac{hb}{(n+a-b)(r+a-b)} \right] \right\}$$

Where

$$(16) (A20) \quad h = \frac{r+a-b}{(a+r)(1-\gamma)} \left\{ \rho - \gamma r - \frac{(\mu-r+\varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\} > 0$$

The RRA coefficient at  $z(t) = \hat{z}$  (line 9 table 1) is calculated using equation (36)

$$(36) (D5) \quad RRA(z = \hat{z}) = (1 - \gamma) \left\{ 1 + \left[ \frac{\hat{z}}{1-\hat{z}} \right] \left[ \frac{h}{(r+a-b)} \right] \right\}$$

The Elasticity of Substitution (=s) at  $z(t) = \hat{z}$  (line 10 table 1) is calculated using equation (37)

$$(37) (D6) \quad s(z = \hat{z}) = \frac{1-\hat{z}}{1-\gamma} \quad \text{when} \quad z(t) = \hat{z}, \mu - r, \text{ and } \sigma^2 \quad \text{held constant}$$

The product of s\*RRA at  $z(t) = \hat{z}$  (line 11 table 1) is calculated using the product of equations (36) and (37) or by using equation (39)

$$(39)(D8) \quad s * RRA \text{ at } [z(t) = \hat{z}] = s(z = \hat{z}) * RRA(z = \hat{z}) = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} \hat{z}$$



**Table 1 results interpretations:**

First, the mean RRA decreases as one moves to the right of the table (3.21 down to 2.18), and approaches the value  $(1 - \gamma) = 1.836$ . The equity premium puzzle may be explained since the model, where the stock price movement is represented using a right skewed non-Gaussian model (the Gumbel distribution), generates the mean and variance of the consumption growth rate with the mean RRA coefficient approaching the value  $(1 - \gamma) = 1.836$ .

Habit persistence reduces the product  $s \cdot \text{RRA}$  extremely below one smoothing consumption growth beyond the smoothing achieved by the life cycle permanent income hypothesis with time separable utility [see equations (40) and (41)]. Equation (56)

$$(56) \quad \frac{\text{var} \left[ \frac{dc(t)}{c(t)} \right]}{dt} = \frac{6(\mu - r + \varepsilon \sigma)^2}{\pi^2 \sigma^2 (1 - \gamma)^2}$$

represent the effect of the consumption smoothing on the equity premium.

Second, the modal value of the state variable ( $= \hat{z}$ ) is about 0.75 for all (a,b) pairs. The model estimates that the subsistence level of consumption generated by habit persistence ( $= x(t)$ ) is about 75% of the consumption level.

Third, the intertemporal elasticity of substitution in consumption ( $= s$ ) is extremely below one (around 0.15).

Forth, the product of the elasticity of substitution and the RRA coefficient ( $= s \cdot \text{RRA} = \text{wedge}$ ) is about 0.26 for the (a,b) pairs that explain the equity premium puzzle with low RRA coefficient. Hence, the habit persistence creates a wedge between the RRA coefficient and the invers of the elasticity substitution ( $= 1/s$ ) [as it appears in equation (32.5)] and thus may explain the equity premium puzzle.

In summary, Table 1 shows that habit persistence can generate the sample mean and variance of the consumption growth rate with low risk aversion, hence giving a plausible explanation to the equity premium puzzle.

**VI) The effect of time separability in utility preferences on the Equity Premium puzzle**

To show that time separability in utility preferences causes the equity premium puzzle observed in Hehra and Prescott (1985) paper, I will follow the methodology used by Constantinides (1990).

I define  $m_{t+1}$  as the marginal rate of substitution at  $t+1$  [different than the constant  $m$  defined in equation (15) and equation (A19)].

$R_{ft} = 1 + r_{ft}$  where  $r_{ft}$  is the riskless rate of interest between periods  $t$  and  $t+1$ .

$R_t = 1 + r_t$  where  $r_t$  is the rate of return between periods  $t$  and  $t+1$ .

The Euler equation is

$$(64) \quad E[m_{t+1}R_{Ft} | I_t] = 1$$

Where  $I_t$  is the public information in period t.

Since at time t  $R_{Ft}$  is known (i.e. in the information set  $I_t$ ) then

$$E[m_{t+1}R_{Ft} | I_t] = R_{Ft} E[m_{t+1} | I_t] = 1$$

Hence

$$(65) \quad E[m_{t+1} | I_t] = (R_{Ft})^{-1}$$

Applying the expectation operator on both sides of equation (65) yields

$$(66) \quad E(m) = E[(R_F)^{-1}]$$

Utilizing the Jensen's inequality on equation (66) I get

$$(67) \quad E(m) = E[(R_F)^{-1}] \geq [E(R_F)]^{-1}$$

Apply the Euler equation using  $R_{t+1}$  (i.e. one plus the rate of return) thus

$$(68) \quad E[m_{t+1}R_{t+1} | I_t] = 1$$

Applying the expectation operator on both sides of equation (68) yields

$$(69) \quad E[mR] = 1$$

Following the methodology of Hansen and Jagannathan (1988) I get

$$(70) \quad 1 = E[mR] = E(m)E(R) + cov(m, R)$$

from covariance definition:  $cov(m, R) = \rho std(m) std(R)$ ,  $-1 < \rho < 1$

using  $\rho = -1$  yields the minimum of the covariance expression, so when inserting this into equation (70) I get

$$(70) \quad 1 = E[mR] = E(m)E(R) + cov(m, R) \geq E(m)E(R) - std(m) std(R)$$

Inserting equation (67) into (70) yields

$$(70) \quad 1 = E[mR] = E(m)E(R) + cov(m, R) \geq E(m)E(R) - std(m) std(R)$$

$$\geq \frac{E(R)}{E(R_F)} - \text{std}(m) \text{std}(R)$$

Thus 
$$1 \geq \frac{E(R)}{E(R_F)} - \text{std}(m) \text{std}(R)$$

Solving for  $\text{std}(m)$

$$(71) \quad \text{std}(m) \geq \frac{\left[ \frac{E(R)}{E(R_F)} - 1 \right]}{\text{std}(R)}$$

Mehra and Prescott (1985) use the following discrete utility form

$$(72) \quad u(c_t) = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{\gamma} \right) (c_t)^\gamma$$

The marginal rate of substitution is calculated

$$(73) \quad m_{t+1} = \frac{\left( \frac{\partial u}{\partial c_{t+1}} \right)}{\left( \frac{\partial u}{\partial c_t} \right)} = \frac{\beta^{t+1} \left( \frac{1}{\gamma} \right)^\gamma (c_{t+1})^{\gamma-1}}{\beta^t \left( \frac{1}{\gamma} \right)^\gamma (c_t)^{\gamma-1}} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\gamma-1}$$

Assuming the consumption growth rate is bounded such that

$$(74) \quad g_1 \leq \frac{c_{t+1}}{c_t} \leq g_2$$

Or 
$$\frac{1}{g_2} \leq \frac{c_t}{c_{t+1}} \leq \frac{1}{g_1}$$

Since both  $\beta$  and  $(1 - \gamma)$  are positive, applying on both sides of the inequality yields

$$\beta \left( \frac{1}{g_2} \right)^{1-\gamma} \leq \beta \left( \frac{c_t}{c_{t+1}} \right)^{1-\gamma} \leq \beta \left( \frac{1}{g_1} \right)^{1-\gamma}$$

Inserting equation (73) above yields

$$(75) \quad \beta (g_2)^{\gamma-1} \leq m_{t+1} \leq \beta (g_1)^{\gamma-1}$$

Since equation (75) defines the upper and lower bounds of  $m_{t+1}$  the standard deviation is

$$(76) \quad \text{stdev}(m) \leq \frac{\beta (g_1)^{\gamma-1} - \beta (g_2)^{\gamma-1}}{2}$$

Using both equation (76) and (71) I get

$$(77) \quad \frac{\left[ \frac{E(R)}{E(R_F)} - 1 \right]}{std(R)} \leq stv(m) \leq \frac{\beta(g_1)^{\gamma-1} - \beta(g_2)^{\gamma-1}}{2}$$

Hence

$$(78) \quad \frac{2 \left[ \frac{E(R)}{E(R_F)} - 1 \right]}{std(R)} \leq \beta(g_1)^{\gamma-1} - \beta(g_2)^{\gamma-1}$$

Mehra and Prescott (1985) used their 1889-1978 data sample to make the following estimates:

$$E(R_F) = 1.01 \text{ per year}, \quad E(R) = 1.07 \text{ per year}, \quad std(R) = 0.165 \text{ per year}, \quad g_1 = 0.982, \\ g_2 = 1.054,$$

Inserting the above estimates into equation (78) I get

$$(79) \quad (0.982)^{\gamma-1} - (1.054)^{\gamma-1} \geq \frac{0.72}{\beta}$$

Solving for  $(1 - \gamma)$  and inserting specific  $\beta$  values I can calculate the level of risk aversion in each case.

From equation (58) I get  $RRA > (1 - \gamma)$ , hence calculating  $(1 - \gamma)$  levels represent the lower bound of RRA coefficient (= relative risk aversion coefficient).

From equation (79) I find: for  $\beta = 0.8$  the  $(1 - \gamma) = 16$ , for  $\beta = 0.9$  the  $(1 - \gamma) = 14$ , for  $\beta = 1$  the  $(1 - \gamma) = 12$ , for  $\beta = 1.1$  the  $(1 - \gamma) = 11$ , for  $\beta = 1.2$  the  $(1 - \gamma) = 10$ .

One can see that the levels of  $(1 - \gamma)$  for a time separable (=discrete) model ( $10 \leq 1 - \gamma \leq 16$ ) are much higher than the levels in a nonsaparable model as seen in table 1 ( $1 - \gamma = 1.836$ ). This demonstrates that time separability in utility preferences creates the equity premium puzzle observed by Mehra and Prescott (1985).

The issue is that the lower bound on the consumption growth rate ( $=g_1$ ) affects the upper bound on the marginal rate of substitution [=  $m_{t+1}$ , see equation (75)]. Reitz (1988) also shows the effect of lower bound on consumption growth on the equity premium puzzle.

## **VII) The effect of time separability in utility preferences and Habit persistence on the Equity Premium puzzle**

Taking into consideration both time separability in utility preferences and Habit persistence, I will redo the calculations performed in part (VI).

Equation (71) is derived in the same manner as in part VI.

$$(71) \quad std(m) \geq \frac{\left[ \frac{E(R)}{E(R_F)} - 1 \right]}{std(R)}$$

Will now use a discrete utility function that includes Habit persistence

$$(80) \quad u(c_t, x_t) = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{\gamma} \right) (c_t - x_t)^\gamma$$

Where  $x_t$  represent the subsistence level of consumption generated by Habit Persistence.

Inserting equation (22) into (80)

$$(22) \quad z(t) = \frac{x(t)}{c(t)} \quad \text{i.e.} \quad x(t) = z(t)c(t) \quad \text{I derive}$$

$$(81) \quad u(c_t, x_t) = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{\gamma} \right) \{[1 - z_t]c_t\}^\gamma$$

Deriving the marginal rate of substitution for this utility

$$(82) \quad m_{t+1} = \frac{\left( \frac{\partial u}{\partial c_{t+1}} \right)}{\left( \frac{\partial u}{\partial c_t} \right)} = \frac{\beta^{t+1} \left( \frac{1}{\gamma} \right)^\gamma (c_{t+1})^{\gamma-1} (1-z_{t+1})^\gamma}{\beta^t \left( \frac{1}{\gamma} \right)^\gamma (c_t)^{\gamma-1} (1-z_t)^\gamma} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\gamma-1} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Assuming the consumption growth rate is bounded such that

$$(74) \quad g_1 \leq \frac{c_{t+1}}{c_t} \leq g_2$$

Or

$$\frac{1}{g_2} \leq \frac{c_t}{c_{t+1}} \leq \frac{1}{g_1}$$

Since both  $\beta$ ,  $(1 - \gamma)$  and  $(1 - z_t)$  are positive, applying on both sides of the inequality yields

$$\beta \left( \frac{1}{g_2} \right)^{1-\gamma} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma \leq \beta \left( \frac{c_{t+1}}{c_t} \right)^{\gamma-1} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma \leq \beta \left( \frac{1}{g_1} \right)^{1-\gamma} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Inserting equation (82) above yields

$$(83) \quad \beta (g_2)^{\gamma-1} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma \leq m_{t+1} \leq \beta (g_1)^{\gamma-1} \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Since equation (83) defines the upper and lower bounds of  $m_{t+1}$  the standard deviation is

$$(84) \quad stv(m) \leq \left[ \frac{\beta(g_1)^{\gamma-1} - \beta(g_2)^{\gamma-1}}{2} \right] \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Using both equation (84) and (71) I get

$$(85) \quad \frac{\left[ \frac{E(R)}{E(R_F)} - 1 \right]}{std(R)} \leq stv(m) \leq \left[ \frac{\beta(g_1)^{\gamma-1} - \beta(g_2)^{\gamma-1}}{2} \right] \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Hence

$$(86) \quad \frac{2 \left[ \frac{E(R)}{E(R_F)} - 1 \right]}{std(R)} \leq [\beta(g_1)^{\gamma-1} - \beta(g_2)^{\gamma-1}] \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma$$

Mehra and Prescott (1985) used their 1889-1978 data sample to make the following estimates:

$$E(R_F) = 1.01 \text{ per year}, \quad E(R) = 1.07 \text{ per year}, \quad std(R) = 0.165 \text{ per year}, \quad g_1 = 0.982, \\ g_2 = 1.054,$$

Inserting the above estimates into equation (86) I get

$$(87) \quad \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma [(0.982)^{\gamma-1} - (1.054)^{\gamma-1}] \geq \frac{0.72}{\beta}$$

$$\text{Or} \quad \left( \frac{1-z_{t+1}}{1-z_t} \right)^\gamma \left[ \frac{1}{(0.982)^{1-\gamma}} - \frac{1}{(1.054)^{1-\gamma}} \right] \geq \frac{0.72}{\beta}$$

Using the same values of  $\beta$  used in part VI, the right hand side of equation (87) will have the same values as in part VI.

Assuming increase in consumption over time (i.e.  $c_{t+1} > c_t$ ), I can assume a decrease in subsistence rate of consumption generated by Habit persistence (i.e.  $z_{t+1} < z_t$ ) in order to keep the subsistence level of consumption generated by Habit persistence fixed (i.e.  $x_{t+1} \cong x_t$ ). Thus,  $\frac{1-z_{t+1}}{1-z_t} > 1$ . Hence, for the same values of  $\beta$  I need lower levels of  $(1-\gamma)$  to meet equation (87).

This shows that adding Habit Persistence to a utility function will reduce levels of risk aversion (i.e. RRA) since  $(1-\gamma)$  is the lower bound of the RRA.

## VIII) Conclusion

My results suggest that habit persistence can generate the Mehra and Prescott's (1985) sample mean and variance of the consumption growth rate with low risk aversion, hence giving a plausible explanation to the equity premium puzzle.

The habit persistence creates a wedge between the RRA coefficient and the invers of the elasticity substitution ( $=1/s$ ) [as it appears in equation (32.5)] and thus may explain the equity premium puzzle.

Habit persistence reduces the product  $s \cdot RRA$  extremely below one, thus smoothing consumption growth (see equations 40 and 41) beyond the smoothing achieved by the life cycle permanent income hypothesis with time separable utility. From equation (56), such smoothing of consumption growth (i.e. reduction in  $\frac{var[\frac{dc(t)}{c(t)}]}{dt}$ ) will reduce the equity premium (by reducing  $\mu - r + \varepsilon\sigma$ ), hence may explain the equity premium puzzle. Furthermore, I show that time separability in utility preferences (i.e.  $b=0$ ) creates the equity premium puzzle (i.e. very high equity premium) presented in the Mehra and Prescott (1985) paper. Remember, Mehra and Prescott use a discrete-time economy in which the state variable is determined by a Markov process with two realization. Thus, their model present high time separability that generates the equity premium puzzle. My model represent a continuous time economy in which the forcing process is a diffusion, thus present a nonseparable utility ( $b>0$  in table 1) which reduces the equity premium thus explain the equity premium puzzle.

The model in this paper incorporates habit persistence in utility which causes low risk aversion (minimal unconditional mean of the RRA coefficient = 2.18, since low  $1 - \gamma$  means low RRA) which generates low variability in consumption growth rate [mean value around  $(0.036)^2$  and see equations 40 and 41] and low variability in the marginal rate of substitution in consumption ( $m_t$  see equations 73 and 82). Since the modal value of the subsistence rate of consumption generated by Habit Persistence ( $= z$ ) is around 0.75 in my model, a high value, the subsistence level of consumption ( $= x$ ) is about 75% of the normal consumption rate. Thus, a small decrease in consumption causes a big decrease in consumption net of the subsistence level ( $= c - x$ ) thus causing a big decrease in the marginal rate of substitution ( $= m_t$  see equation 73 and 82). Such low risk aversion can explain the observed equity premium.

The results displayed in the paper show that a rational expectations model with time nonseparability of preferences, with habit persistence and with stock price movement represented by a right skewed non-Gaussian model (the Gumbel distribution) can explain the equity premium puzzle. The model is able to find acceptable relative risk aversion (RRA) of the representative agent to match the sample estimates.

Habit persistence differs from the known state and time-separable preferences. Habit persistence should be embedded in new models of the business cycle, labor behavior, public finance and dynamics.

## Appendices:

### **Appendix A: Proof of Theorem 1**

I use a technique used by Davis and Norman (1987). If  $c(t)$  and  $\alpha(t)$  are not optimal at  $t$ , I assume an optimal policy  $c^*(s)$  and  $\alpha^*(s)$  for  $s \geq t$ .

Equation (6)

$$(6) \quad dW_t = \{[(\mu - r + \varepsilon\sigma)\alpha_t + r]W_t - c_t\}dt + \left(\frac{\pi}{\sqrt{6}}\right)\alpha_t W_t \sigma dw_t$$

Becomes

$$(A1) \quad dW(s) = \{[(\mu - r + \varepsilon\sigma)\alpha^*(s) + r]W(s) - c^*(s)\}ds + \left(\frac{\pi}{\sqrt{6}}\right)\alpha^*(s)W(s)\sigma dw(s)$$

Equation (11) above

$$(11) \quad bc(t) - ax(t) = 0$$

Becomes

(A2)

$$bc^*(s) - ax(s) = \frac{dx(s)}{ds}$$

or

$$dx(s) = [bc^*(s) - ax(s)]ds$$



Equation (8)

$$(8) \quad c(t) - x(t) = (r + a - b) \left[ W(t) - \frac{x(t)}{r+a-b} \right]$$

Becomes

$$(A3) \quad c^*(s) - x(s) = h \left[ W(s) - \frac{x(s)}{r+a-b} \right]$$

Next I will derive

$$d \left[ W(s) - \frac{x(s)}{r+a-b} \right] = dW(s) - \frac{dx(s)}{r+a-b}$$

I will insert equation (A1) and (A2) into the above expression to derive (A4)

$$(A4) \quad d \left[ W(s) - \frac{x(s)}{r+a-b} \right] = \{[(\mu - r + \varepsilon\sigma)\alpha^*(s) + r]W(s) - c^*(s)\}ds + \left(\frac{\pi}{\sqrt{6}}\right)\alpha^*(s)W(s)\sigma dw(s) -$$

$$\begin{aligned} & - \frac{[bc^*(s) - ax(s)]ds}{r+a-b} = \\ & = \left\{[(\mu - r + \varepsilon\sigma)\alpha^*(s) + r]W(s) - c^*(s) - \frac{[bc^*(s) - ax(s)]}{r+a-b}\right\}ds + \left(\frac{\pi}{\sqrt{6}}\right)\sigma\alpha^*(s)W(s)dw(s) = \end{aligned}$$

I will define m such that:

$$\alpha^*(s)W(s) = m \left[ W(s) - \frac{x(s)}{r+a-b} \right]$$

Hence

$$(A5) \quad \alpha^*(s) = m \left[ 1 - \frac{\frac{x(s)}{W(s)}}{r+a-b} \right]$$

Inserting  $\alpha^*(s)$  into (A4) yield

$$(A4) \quad d \left[ W(s) - \frac{x(s)}{r+a-b} \right] = \left\{ \left[ (\mu - r + \varepsilon\sigma)m \left[ 1 - \frac{\frac{x(s)}{W(s)}}{r+a-b} \right] + r \right] W(s) - \frac{a[c^*(s) - x(s)]}{r+a-b} - \frac{[c^*(s)r]}{r+a-b} \right\} ds +$$

$$+ \left(\frac{\pi}{\sqrt{6}}\right)\sigma m \left[ W(s) - \frac{x(s)}{r+a-b} \right] dw(s) =$$

Inserting equation (A3) into equation (A4) yields

$$(A4) = \left\{ \left[ (\mu - r + \varepsilon\sigma)m \left[ 1 - \frac{x(s)}{W(s)} \right] + r \right] W(s) - \frac{ah \left[ W(s) - \frac{x(s)}{r+a-b} \right]}{r+a-b} - \frac{[x(s)+h \left[ W(s) - \frac{x(s)}{r+a-b} \right]]}{r+a-b} \right\} ds + \\ + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m \left[ W(s) - \frac{x(s)}{r+a-b} \right] dw(s) =$$

Arriving at

$$(A4) \frac{d \left[ W(s) - \frac{x(s)}{r+a-b} \right]}{\left[ W(s) - \frac{x(s)}{r+a-b} \right]} = \left\{ (\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b} \right\} ds + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(s) = nds + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(s)$$

Where i define

$$n = (\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b}$$

Since

$$d \ln \left[ W(s) - \frac{x(s)}{r+a-b} \right] = \frac{d \left[ W(s) - \frac{x(s)}{r+a-b} \right]}{\left[ W(s) - \frac{x(s)}{r+a-b} \right]}$$

Equation (A4) becomes

$$(A6) \quad d \ln \left[ W(s) - \frac{x(s)}{r+a-b} \right] = nds + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(s)$$

Using exponent on sides yield

$$e^{\left\{ d \ln \left[ W(s) - \frac{x(s)}{r+a-b} \right] \right\}} = e^{\{nds\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(s) \right\}}$$

Hence

$$d \left[ W(s) - \frac{x(s)}{r+a-b} \right] = e^{\{nds\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(s) \right\}}$$

Solving the above differential equation yields the following solution

$$(A7) \quad W(s) - \frac{x(s)}{r+a-b} = \left[ W(t) - \frac{x(t)}{r+a-b} \right] e^{\{n(s-t)\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right\}} \quad , s \geq t$$

I define the state valuation function,  $V(.,.)$ , as a function of the two variables,  $W(t)$  and  $x(t)$ . Thus, the maximization problem is:

(A8)

$$V[W(t), x(t)] = \max E_t \int_t^{\infty} e^{-\rho(s-t)} u[c(s), x(s)] ds = \max E_t \int_t^{\infty} e^{-\rho(s-t)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds$$

where  $\rho$  represents the constant utility discount rate, and  $E_t$  represents the operator for expectations conditional on the information set at time  $t$ .

By inserting the optimal values  $c^*(s)$  and  $\alpha^*(s)$ , for  $s \geq t$ , the maximum achieved, thus

(A9)

$$V[W(s), x(s)] = E_t \int_s^{\infty} e^{-\rho(s-t)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds =$$

By inserting equation (A3) into equation (A9) I derive

$$\begin{aligned} V[W(s), x(s)] &= E_t \int_s^{\infty} e^{-\rho(s-t)} \frac{1}{\gamma} (h)^\gamma \left[ W(s) - \frac{x(s)}{r+a-b} \right]^\gamma ds = \\ &= \int_s^{\infty} e^{-\rho(s-t)} E_t \left\{ \frac{1}{\gamma} (h)^\gamma \left[ W(s) - \frac{x(s)}{r+a-b} \right]^\gamma \right\} ds \end{aligned}$$

Concentrating on developing the expression within the integral and inserting equation (A7) yields

$$\begin{aligned} &e^{-\rho(s-t)} E_t \left\{ \frac{1}{\gamma} (h)^\gamma \left[ W(s) - \frac{x(s)}{r+a-b} \right]^\gamma \right\} = \\ &= e^{-\rho(s-t)} E_t \left\{ \frac{1}{\gamma} (h)^\gamma \left[ \left[ W(t) - \frac{x(t)}{r+a-b} \right] e^{\{n(s-t)\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right\}} \right]^\gamma \right\} = \\ &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{-\rho(s-t)} [e^{n(s-t)}]^\gamma E_t \left\{ \left[ e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right\}} \right]^\gamma \right\} = \\ &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{-\rho(s-t)} [e^{n(s-t)}]^\gamma E_t \left\{ \left[ e^{\left\{ \frac{\gamma}{2} \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right\}} \right]^2 \right\} = \end{aligned}$$

$$= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{-\rho(s-t)} [e^{n(s-t)}]^\gamma e^{\left\{ E_t \left[ \frac{\gamma}{2} \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right]^2 \right\}} =$$

Since with a Wiener process  $E_t[w(s)] = E_t[w(t)] = 0$  thus

$$= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{-\rho(s-t)} [e^{n(s-t)}]^\gamma e^{\left\{ \text{var} \left[ \frac{\gamma}{2} \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s) - w(t)] \right] \right\}} =$$

Since with a Wiener process  $\text{var}[w(s)] = s$  and  $\text{var}[w(t)] = t$  thus

$$\begin{aligned} &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{-\rho(s-t)} [e^{n(s-t)}]^\gamma e^{\left\{ \left( \frac{\gamma^2}{4} \right) \left( \frac{\pi^2}{6} \right) (m^2) (\sigma^2) (s-t) \right\}} = \\ &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{\left\{ \left[ -\rho + \gamma n + \left( \frac{\pi^2}{6} \right) \left( \frac{\gamma^2 m^2 \sigma^2}{4} \right) \right] (s-t) \right\}} = \\ &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{\left\{ \left[ \rho - \gamma n - \left( \frac{\pi^2}{6} \right) \left( \frac{\gamma^2 m^2 \sigma^2}{4} \right) \right] (t-s) \right\}} = \end{aligned}$$

I define

$$H = \rho - \gamma n - \left( \frac{\pi^2}{6} \right) \left( \frac{\gamma^2 m^2 \sigma^2}{4} \right)$$

Thus equation (A9) becomes

$$\begin{aligned} V[W(s), x(s)] &= \int_s^\infty \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma e^{\{H(t-s)\}} ds = \\ &= \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma \int_s^\infty e^{\{H(t-s)\}} ds = \end{aligned}$$

Solving the integral yields equation (A10)

$$(A10) \quad V[W(s), x(s)] = \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma \left( \frac{-1}{H} \right)$$

Where

$$H = \rho - \gamma n - \left(\frac{\pi^2}{6}\right) \left(\frac{\gamma^2 m^2 \sigma^2}{4}\right)$$

$$(A11) \quad n = (\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b}$$

I now define M(t) as follows

$$\begin{aligned} M(t) &= \int_0^t e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + \int_t^\infty e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds = \\ &= \int_0^t e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + e^{-\rho t} \int_t^\infty e^{-\rho(s-t)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds = \end{aligned}$$

Inserting equation (A9) into the above equation to derive equation (A12)

$$(A12) \quad M(t) = \int_0^t e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + e^{-\rho t} V[W(t), x(t)]$$

I now define the derivative

$$(A13) \quad N(t) = \frac{dM(t)}{dt} = e^{-\rho t} \left(\frac{1}{\gamma}\right) [c(t) - x(t)]^\gamma + (-\rho)e^{-\rho t} V[W(t), x(t)] + e^{-\rho t} \left[\frac{dV[W(t), x(t)]}{dt}\right]$$

I use Ito's Lemma to calculate  $\frac{dV[W(t), x(t)]}{dt}$

The closed form problem is defined below:

$$(A10) \quad V[W(t), x(t)] = \frac{(h)^\gamma}{\gamma} \left[W(t) - \frac{x(t)}{r+a-b}\right]^\gamma \left(\frac{-1}{H}\right) = \max E_t \int_t^\infty e^{-\rho(s-t)} \frac{1}{\gamma} [c(t) - x(t)]^\gamma ds$$

Where

$$(A14) \quad H = \rho - \gamma \left[(\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b}\right] - \left(\frac{\pi^2}{6}\right) \left(\frac{\gamma^2 m^2 \sigma^2}{4}\right)$$

$$(6) \quad dW(t) = \{[\alpha_t(\mu - r + \varepsilon\sigma) + r]W(t) - c_t\}dt + \left(\frac{\pi}{\sqrt{6}}\right) \alpha_t W(t) \sigma dw_t$$

$$(A2) \quad dx(t) = [bc^*(t) - ax(t)]dt + 0dw_t$$

Applying Ito's Lemma formulation:

$$dV(W_t, x_t) = \frac{\partial V}{\partial W_t} dW_t + \frac{\partial V}{\partial x_t} dx_t + \frac{1}{2} \left( \frac{\partial^2 V}{\partial W_t^2} \right) (dW_t)^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial x_t^2} \right) (dx_t)^2 + \left( \frac{\partial^2 V}{\partial W_t \partial x_t} \right) (dW_t)(dx_t)$$

By dividing both side of the equation by  $dt$  I get:

$$(A15) \frac{dV}{dt} = \left( \frac{\partial V}{\partial W_t} \right) \left( \frac{dW_t}{dt} \right) + \left( \frac{\partial V}{\partial x_t} \right) \left( \frac{dx_t}{dt} \right) + \frac{1}{2} \left( \frac{\partial^2 V}{\partial W_t^2} \right) \left( \frac{(dW_t)^2}{dt} \right) + \frac{1}{2} \left( \frac{\partial^2 V}{\partial x_t^2} \right) \left( \frac{(dx_t)^2}{dt} \right) + \left( \frac{\partial^2 V}{\partial W_t \partial x_t} \right) \frac{(dW_t)(dx_t)}{dt}$$

Solving the expressions using equations (6), (A2) and (A10):

$$\frac{dV}{dt} = \rho \max E_t \int_t^{\infty} e^{-\rho(s-t)} \frac{1}{\gamma} [c(t) - x(t)]^\gamma ds = \rho V$$

$$dW(t)dx(t) = 0 \text{ [Since } (dt)^2 = dt dw_t = 0 \text{ and } dw_t dw_t = dt \text{]}$$

$$(dW_t)^2 = \left( \frac{\pi^2}{6} \right) \alpha^2 \sigma^2 (W_t)^2 dt$$

$$(dx_t)^2 = 0$$

Thus equation (A15) is

$$(A15) \frac{dV}{dt} = V_W \{ [\alpha_t(\mu - r + \varepsilon\sigma) + r]W(t) - c_t \} + V_x \{ bc^*(t) - ax(t) \} + \frac{1}{2} V_{WW} \left\{ \left( \frac{\pi^2}{6} \right) \alpha^2 \sigma^2 (W_t)^2 \right\}$$

I will, next, perform first order conditions on  $N(t)$  [equation (A13)]:

$$(A13) N(t) = e^{-\rho t} \left( \frac{1}{\gamma} \right) [c(t) - x(t)]^\gamma + (-\rho) e^{-\rho t} V[W(t), x(t)] + e^{-\rho t} \left[ \frac{dV[W(t), x(t)]}{dt} \right]$$

Where

$$(A10) V[W(t), x(t)] = \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma \left( \frac{-1}{H} \right)$$

The derivatives of equation (A10)

$$V_W = \frac{(h)^\gamma}{\gamma} \gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right) = (h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right)$$

$$\begin{aligned}
V_{WW} &= (h)^\gamma (\gamma - 1) \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-2} \left( \frac{-1}{H} \right) \\
V_x &= \frac{(h)^\gamma}{\gamma} \gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{r+a-b} \right) \left( \frac{-1}{H} \right) = \\
&= (h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{r+a-b} \right) \left( \frac{-1}{H} \right)
\end{aligned}$$

FOC1: with respect to  $c(t)$

$$0 = \frac{dN(t)}{dc(t)} = e^{-\rho t} \left( \frac{1}{\gamma} \right) \gamma [c(t) - x(t)]^{\gamma-1} + 0 + e^{-\rho t} \{V_W(-1) + V_x b + 0\}$$

Thus

$$(A16) \quad 0 = [c(t) - x(t)]^{\gamma-1} - V_W + V_x b$$

Insert the derivatives into equation (A16):

$$0 = [c(t) - x(t)]^{\gamma-1} - (h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right) + b(h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{r+a-b} \right) \left( \frac{-1}{H} \right)$$

Solving for  $c(t)$

$$(A17) \quad c^*(t) = x(t) - (h)^{\frac{\gamma}{\gamma-1}} \left[ W(t) - \frac{x(t)}{r+a-b} \right] \left[ \frac{(r+a)}{H(r+a-b)} \right]$$

FOC2: with respect to  $\alpha(t)$

$$0 = \frac{dN(t)}{d\alpha(t)} = 0 + 0 + e^{-\rho t} \left\{ V_W(\mu - r + \varepsilon\sigma)W(t) + \frac{1}{2} V_{WW} \left[ \left( \frac{\pi^2}{6} \right) 2\alpha\sigma^2 (W_t)^2 \right] \right\}$$

Thus

$$(A18) \quad 0 = V_W(\mu - r + \varepsilon\sigma)W(t) + V_{WW} \left[ \left( \frac{\pi^2}{6} \right) \alpha\sigma^2 (W_t)^2 \right]$$

Insert the derivatives into equation (A18):

$$0 = (h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right) (\mu - r + \varepsilon\sigma)W(t) + \\ + (h)^\gamma (\gamma - 1) \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-2} \left( \frac{-1}{H} \right) \left[ \left( \frac{\pi^2}{6} \right) \alpha \sigma^2 (W_t)^2 \right]$$

Solve for  $\alpha(t)$

$$\alpha^*(t) = \left[ \frac{(\mu - r + \varepsilon\sigma)}{\left( \frac{\pi^2}{6} \right) \sigma^2 W(t) (1 - \gamma)} \right] \left[ W(t) - \frac{x(t)}{r+a-b} \right]$$

Insert equation (A5) into the above equation

$$(A5) \quad \alpha^*(t) = m \left[ 1 - \frac{\frac{x(t)}{W(t)}}{r+a-b} \right] \\ m \left[ 1 - \frac{\frac{x(t)}{W(t)}}{r+a-b} \right] = \left[ \frac{(\mu - r + \varepsilon\sigma)}{\left( \frac{\pi^2}{6} \right) \sigma^2 W(t) (1 - \gamma)} \right] \left[ W(t) - \frac{x(t)}{r+a-b} \right]$$

Solve for m

$$(A19) \quad m = \frac{(\mu - r + \varepsilon\sigma)}{\left( \frac{\pi^2}{6} \right) \sigma^2 (1 - \gamma)}$$

Where  $0 \leq m \leq 1$

Insert equation (A19) into equation (A14)

$$(A14) \quad H = \rho - \gamma \left[ (\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b} \right] - \left( \frac{\pi^2}{6} \right) \left( \frac{\gamma^2 m^2 \sigma^2}{4} \right) = \\ = \rho - \gamma \left\{ (\mu - r + \varepsilon\sigma) \left[ \frac{(\mu - r + \varepsilon\sigma)}{\left( \frac{\pi^2}{6} \right) \sigma^2 (1 - \gamma)} \right] + r - \frac{h(a+r)}{r+a-b} \right\} -$$



$$-\left(\frac{\pi^2}{6}\right) \left( \frac{\gamma^2 \left[ \frac{(\mu - r + \varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right) \sigma^2 (1 - \gamma)} \right]^2 \sigma^2}{4} \right) =$$

$$H = \rho - \gamma r - \frac{\gamma(\mu - r + \varepsilon\sigma)^2(4 - 3\gamma)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1 - \gamma)^2} + \frac{\gamma(a + r)h}{r + a - b} =$$

I define

$$(A20) \quad h = \frac{r+a-b}{(a+r)(1-\gamma)} \left\{ \rho - \gamma r - \frac{(\mu-r+\varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

thus

$$H = \rho - \gamma r - \frac{\gamma(\mu - r + \varepsilon\sigma)^2(4 - 3\gamma)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1 - \gamma)^2} +$$

$$+ \left[ \frac{\gamma(a + r)}{r + a - b} \right] \left[ \frac{r + a - b}{(a + r)(1 - \gamma)} \right] \left\{ \rho - \gamma r - \frac{(\mu - r + \varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1 - \gamma)} \right\} =$$

Hence

$$(A21) \quad H = \left(\frac{1}{1-\gamma}\right) \left\{ \rho - \gamma r - \frac{9\gamma(\mu-r+\varepsilon\sigma)^2}{2(\pi^2)\sigma^2(1-\gamma)} \right\}$$

Using equations (A11), (A20) and (A19) I will derive  $n$

$$(A11) \quad n = (\mu - r + \varepsilon\sigma)m + r - \frac{h(a+r)}{r+a-b} =$$

$$= (\mu - r + \varepsilon\sigma) \left[ \frac{(\mu - r + \varepsilon\sigma)}{\left(\frac{\pi^2}{6}\right)\sigma^2(1 - \gamma)} \right] + r - \frac{r + a - b}{(a + r)(1 - \gamma)} \left\{ \rho - \gamma r - \frac{(\mu - r + \varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1 - \gamma)} \right\} \frac{(a + r)}{r + a - b} =$$

thus

$$(A22) \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

Using equations (A10), (A20) and (A21)

$$(A10) \quad V[W(t), x(t)] = \frac{(h)^y}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^y \left( \frac{-1}{H} \right) =$$

thus

$$(A23) \quad V[W(t), x(t)] = \frac{\left\{ \frac{r+a-b}{(a+r)(1-\gamma)} \left[ \rho - \gamma r - \frac{(\mu-r+\varepsilon\sigma)^2(-1)}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right] \right\}^y}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^y *$$

$$* \left\{ \frac{-1}{\left( \frac{1}{1-\gamma} \right) \left[ \rho - \gamma r - \frac{9\gamma(\mu-r+\varepsilon\sigma)^2}{2(\pi^2)\sigma^2(1-\gamma)} \right]} \right\} =$$

From equation (A7)

$$(A7) \quad W(s) - \frac{x(s)}{r+a-b} = \left[ W(t) - \frac{x(t)}{r+a-b} \right] e^{\{n(s-t)\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(s)-w(t)] \right\}} \quad , s \geq t$$

For inserting  $s = t$  and  $t = 0$  in equation (A7) I derive

$$W(t) - \frac{x(t)}{r+a-b} = \left[ W(0) - \frac{x(0)}{r+a-b} \right] e^{\{n(t-0)\}} e^{\left\{ \left( \frac{\pi}{\sqrt{6}} \right) \sigma m [w(t)-w(0)] \right\}}$$

Since for a Wiener process  $w(0) = 0$ , I derive the wealth (=capital)

$$(A24) \quad W(t) = \frac{x(t)}{r+a-b} + \left[ W(0) - \frac{x(0)}{r+a-b} \right] e^{\left[ nt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m w(t) \right]}$$

where

$$(A22) \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

Next, I will derive the consumption growth rate  $\frac{dc(t)}{c(t)}$

Insert equation (A24) into (A3)

$$(A25) \quad c(t) - x(t) = h \left[ W(t) - \frac{x(t)}{r+a-b} \right] = h \left\{ \left[ W(0) - \frac{x(0)}{r+a-b} \right] e^{\left[ nt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m w(t) \right]} \right\}$$

Apply natural logarithm (Ln) on both sides of equation (A25)

$$(A26) \quad \text{Ln}[c(t) - x(t)] = \text{Ln}(h) + \text{Ln} \left[ W(0) - \frac{x(0)}{r+a-b} \right] + nt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m w(t)$$

Derive equation (A26) by t

$$\frac{d \text{Ln}[c(t) - x(t)]}{dt} = 0 + 0 + n + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m \left[ \frac{dw(t)}{dt} \right]$$

Hence

$$(A27) \quad d \text{Ln}[c(t) - x(t)] = ndt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

Derivative rules determine

$$(A28) \quad d \text{Ln}[c(t) - x(t)] = \frac{d[c(t) - x(t)]}{c(t) - x(t)} = \frac{dc(t) - dx(t)}{c(t) - x(t)}$$

insert equation (A27) into (A28) and derive

$$\frac{dc(t) - dx(t)}{c(t) - x(t)} = ndt + \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

$$(A29) \quad dc(t) - dx(t) = [c(t) - x(t)]ndt + [c(t) - x(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

Insert equation (A2) into (A29)

$$(A2) \quad dx(t) = [bc(t) - ax(t)]dt$$

thus

$$(A29) \quad dc(t) = [bc(t) - ax(t)]dt + [c(t) - x(t)]ndt + [c(t) - x(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

Divide both sides of equation (A29) by  $c(t)$ , and solve for  $\frac{dc(t)}{c(t)}$

$$(A30) \quad \frac{dc(t)}{c(t)} = \left[ n + b - \frac{(n+a)x(t)}{c(t)} \right] dt + \left[ 1 - \frac{x(t)}{c(t)} \right] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

where

$$(A22) \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

I now show that the optimal policies  $c^*(t)$  and  $a^*(t)$  found above are **unique**:

From equation (A12)

$$(A12) \quad M(t) = \int_0^t e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + e^{-\rho t} V[W(t), x(t)]$$

I define the differential

$$(A31) \quad dM(t) = \frac{dM(t)}{dt} dt + \frac{dM(t)}{dW(t)} dW(t)$$

From equation (A13)  $N(t) = \frac{dM(t)}{dt}$  and

$$(A32) \quad \frac{dM(t)}{dW(t)} = e^{-\rho t} V_W$$

Insert equations (A13) and (A32) into (A31) to derive

$$(A33) \quad dM(t) = N(t)dt + e^{-\rho t} V_W dW(t)$$

Thus, for an arbitrary  $(c, \alpha)$ : (A34)  $dM(t) \leq e^{-\rho t} V_W dW(t)$

For the optimal policies  $(c^*, \alpha^*)$  when  $\frac{dM(t)}{dt} = 0$ : (A35)  $dM(t) = e^{-\rho t} V_W dW(t)$

Using equation (A12):

$$(A36) \quad \begin{aligned} M(t = \infty) &= \int_0^{t=\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + e^{-\rho\infty} V[W(t = \infty), x(t = \infty)] \\ &= \int_0^{t=\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + 0 = \int_0^{t=\infty} e^{-\rho s} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds \end{aligned}$$

Applying the expectation operator on both sides of equation (A36) I get

$$(A37)$$

$$E_{t=0}[M(t = \infty)] = E_{t=0} \left\{ \int_0^{t=\infty} e^{-\rho s} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds \right\}$$

For t=0  
(A38)

$$\begin{aligned} M(t = 0) &= \int_0^{t=0} e^{-\rho(s-0)} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds + e^{-\rho 0} V[W(t = 0), x(t = 0)] = \\ &= 0 + V[W(t = 0), x(t = 0)] = \end{aligned}$$

Inserting t=0 in equation (A9) I get

(A39)

$$V[W(t = 0), x(t = 0)] = E_{t=0} \int_{t=0}^{\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds$$

Hence

(A40)

$$M(t = 0) = V[W(t = 0), x(t = 0)] = E_{t=0} \int_{t=0}^{\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds$$

Applying the expectation operator on both sides of equation (A40) I get

(A41)

$$\begin{aligned} E_{t=0}[M(t = 0)] &= E_{t=0} \left\{ E_{t=0} \int_{t=0}^{\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds \right\} = \\ &= E_{t=0} \left\{ \int_{t=0}^{\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds \right\} \end{aligned}$$

Comparing the right hand sides of equations (A37) and (A41) the optimal effect  $c(s) \leq c^*(s)$   
I get

(A42)

$$E_{t=0}[M(t = \infty)] = E_{t=0} \left\{ \int_0^{t=\infty} e^{-\rho s} \frac{1}{\gamma} [c(s) - x(s)]^\gamma ds \right\} \leq$$

$$\leq E_{t=0} \left\{ \int_{t=0}^{\infty} e^{-\rho(s-0)} \frac{1}{\gamma} [c^*(s) - x(s)]^\gamma ds \right\} = E_{t=0}[M(t=0)]$$

Thus

$$(A43) \quad E_{t=0}[M(t=\infty)] \leq E_{t=0}[M(t=0)]$$

Which proves that  $M(t)$  is a supermartingale.

Hence equation (A43) is an equality iff  $c(s) = c^*(s)$  and  $\alpha(s) = \alpha^*(s)$  for all  $s, s \geq 0$ . Thus, for  $t > s \geq 0$  the optimal policies found  $(c^*, \alpha^*)$  are unique.

## **Appendix B: Proof of Theorem 2**

I defined above  $z(t)$  the subsistence rate of consumption generated by Habit Persistence.

$$(22) \quad z(t) = \frac{x(t)}{c(t)}$$

Where  $x(t)$  is the subsistence level of consumption generated by Habit Persistence.

I also defined

$$(24) \quad y(t) = \frac{x(t)}{c(t)-x(t)} = \frac{z(t)}{1-z(t)}$$

From equation (22) I can find

$$c(t) = \frac{x(t)}{z(t)}$$

Derive both sides by  $t$  such that

$$\frac{dc(t)}{dt} = \frac{\frac{dx(t)}{dt} z(t) - x(t) \frac{dz(t)}{dt}}{[z(t)]^2}$$

Multiple both sides by  $\frac{dt}{c(t)}$  to derive

$$(B1) \quad \frac{dc(t)}{c(t)} = \frac{dx(t)}{x(t)} - \frac{dz(t)}{z(t)}$$

From equation (A2)

$$(A2) \quad \frac{dx(s)}{ds} = bc^*(s) - ax(s)$$

I can derive

$$\frac{dx(t)}{dt} = bc(t) - ax(t)$$

Hence

$$(B2) \quad \frac{dx(t)}{x(t)} = \left[ b \frac{c(t)}{x(t)} - a \right] dt = \left[ b \frac{1}{z(t)} - a \right] dt$$

From equation (21), (22) and (B1)

$$(21) (A30) \quad \frac{dc(t)}{c(t)} = \left[ n + b - \frac{(n+a)x(t)}{c(t)} \right] dt + \left[ 1 - \frac{x(t)}{c(t)} \right] \left( \frac{\pi}{\sqrt{6}} \right) \sigma mdw(t)$$

$$\frac{dx(t)}{x(t)} - \frac{dz(t)}{z(t)} = [n + b - (n + a)z(t)]dt + [1 - z(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma mdw(t)$$

Insert equation (B2) into the above

$$\left[ b \frac{1}{z(t)} - a \right] dt - \frac{dz(t)}{z(t)} = [n + b - (n + a)z(t)]dt + [1 - z(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma mdw(t)$$

Solve for  $dz(t)$

$$(B3) \quad dz(t) = [b - az(t) - nz(t)][1 - z(t)]dt - z(t)[1 - z(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma mdw(t)$$

From equation (24)

$$(24) \quad y(t) = \frac{z(t)}{1-z(t)}$$

Derive both sides by t

$$\frac{dy(t)}{dt} = \frac{\frac{dz(t)}{dt}[1 - z(t)] - z(t)\frac{d[1 - z(t)]}{dt}}{[1 - z(t)]^2} = \frac{\frac{dz(t)}{dt}[1 - z(t)] - z(t)\left[\frac{-dz(t)}{dt}\right]}{[1 - z(t)]^2}$$

Hence

$$(B4) \quad dz(t) = dy(t)[1 - z(t)]^2$$

Insert equation (B4) into (B3)

$$dy(t)[1 - z(t)]^2 = [b - az(t) - nz(t)][1 - z(t)]dt - z(t)[1 - z(t)]\left(\frac{\pi}{\sqrt{6}}\right)\sigma mdw(t)$$

I Divide both sides by  $[1 - z(t)]^2$

$$dy(t) = \left[ \frac{b}{1 - z(t)} - (n + a)\frac{z(t)}{1 - z(t)} \right] dt - \frac{z(t)}{1 - z(t)}\left(\frac{\pi}{\sqrt{6}}\right)\sigma mdw(t)$$

Inserting equation (24) into the above yields

$$dy(t) = \{b[1 - y(t)] - (n + a)y(t)\}dt - y(t)\left(\frac{\pi}{\sqrt{6}}\right)\sigma mdw(t)$$

$$(B5) \quad dy(t) = \{b - (n + a - b)y(t)\}dt - y(t)\left(\frac{\pi}{\sqrt{6}}\right)\sigma mdw(t)$$

I continue with deriving  $\bar{y}$  and  $\hat{y}$  (the modal value of  $y$ )

The general case of the one dimension (only variable  $t$ ) Fokker-Plank equation states

Given

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t$$

Where  $\mu(x_t, t)$  is the drift coefficient and  $D(x_t, t) = \frac{\sigma^2(x_t, t)}{2}$  is the diffusion coefficient.

For  $p(x_t, t)$ , the probability density of random variable  $x_t$ , the Fokker-Plank equation is

$$\frac{\partial p(x_t, t)}{\partial t} = -\frac{\partial[\mu(x_t, t)p(x_t, t)]}{\partial x} + \frac{\partial^2[D(x_t, t)p(x_t, t)]}{\partial x^2}$$

In my specific case: if the density of  $y(t)$  is  $p_y(y_t, t)$ , given



$$(B5) \quad dy(t) = [b - (n + a - b)y(t)]dt - y(t) \left(\frac{\pi}{\sqrt{6}}\right) \sigma mdw(t)$$

The Fokker-Plank equation is:

$$\frac{\partial p(y_t, t)}{\partial t} = -\frac{\partial\{[b - (n + a - b)y(t)] p_y(y_t, t)\}}{\partial y} + \frac{\partial^2 \left\{ \frac{[-y(t) \left(\frac{\pi}{\sqrt{6}}\right) \sigma m]^2}{2} p_y(y_t, t) \right\}}{\partial y^2}$$

Hence

$$(B6) \quad \frac{\partial p(y_t, t)}{\partial t} = -\frac{\partial\{[b - (n + a - b)y(t)] p_y(y_t, t)\}}{\partial y} + \left(\frac{\pi^2}{2*6}\right) \frac{\partial^2\{[y(t)]^2 \sigma^2 m^2 p_y(y_t, t)\}}{\partial y^2}$$

Since  $p(y_t, t)$  represent a stationary distribution it satisfies  $\frac{\partial p_y(y_t, t)}{\partial t} = 0$

Which make equation (B6) to become the Pearson equation as in equation (B7).

$$(B7) \quad 0 = -\frac{\partial\{[b - (n + a - b)y(t)] p_y(y_t, t)\}}{\partial y} + \left(\frac{\pi^2}{12}\right) \frac{\partial^2\{[y(t)]^2 \sigma^2 m^2 p_y(y_t, t)\}}{\partial y^2}$$

I integrate both sides of equation (B7) by  $y(t)$  to get

$$(B8) \quad 0 = \left(\frac{\pi^2}{12}\right) \frac{\partial\{[y(t)]^2 \sigma^2 m^2 p_y(y_t, t)\}}{\partial y} - [b - (n + a - b)y(t)] p_y(y_t, t)$$

$$0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 \left\{ [y(t)]^2 \frac{\partial p_y(y_t, t)}{\partial y} + 2y(t) p_y(y_t, t) \right\} - [b - (n + a - b)y(t)] p_y(y_t, t)$$

$$(B9) \quad 0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 [y(t)]^2 \frac{\partial p_y(y_t, t)}{\partial y} - \left\{ b - \left[ (n + a - b) + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 \right] y(t) \right\} p_y(y_t, t)$$

I integrate both sides of equation (B9) by  $y(t)$  to get

$$(B10) \quad 0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 \int_0^\infty [y(t)]^2 \frac{\partial p_y(y_t, t)}{\partial y} dy - b \int_0^\infty p_y(y_t, t) dy + \left[ (n + a - b) + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 \right] \int_0^\infty y(t) p_y(y_t, t) dy$$

under normalization requirement

$$\int_0^{\infty} p_y(y_t, t) dy = 1$$

And that the average definition is

$$\int_0^{\infty} y(t) p_y(y_t, t) dy = \bar{y}$$

Equation (B10) becomes

$$0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 \int_0^{\infty} [y(t)]^2 \frac{\partial p_y(y_t, t)}{\partial y} dy - b + \left[ (n + a - b + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2) \right] \bar{y}$$

The remaining integral will be solved using integration by parts thus

$$0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 \left\{ 0 - 2 \int_0^{\infty} y(t) p_y(y_t, t) dy \right\} - b + \left[ (n + a - b + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2) \right] \bar{y}$$

Hence

$$0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 \{0 - 2\bar{y}\} - b + \left[ (n + a - b + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2) \right] \bar{y}$$

Solving for  $\bar{y}$  I derive

$$(B11) \quad \bar{y} = \frac{b}{n+a-b} < \infty$$

the condition  $n + a - b > 0$  assures  $\bar{y} < \infty$ , i.e.  $\bar{y}$  is finite.

The mode of a variable,  $y(t)$ , that is stationary distributed satisfies  $\frac{\partial p_y(y_t, t)}{\partial y_t} = 0$  so equation (B9) is

$$(B9) \quad 0 = \left(\frac{\pi^2}{12}\right) \sigma^2 m^2 [\hat{y}]^2 0 - \left\{ b - \left[ (n + a - b + \left(\frac{\pi^2}{6}\right) \sigma^2 m^2) \right] \hat{y} \right\} p_y(\hat{y}, t)$$

Solving for  $\hat{y}$  I derive

$$(B12) \quad \hat{y} = \frac{b}{n+a-b+\left(\frac{\pi^2}{6}\right)\sigma^2m^2} < \infty$$

the condition  $n + a - b + \left(\frac{\pi^2}{6}\right)\sigma^2m^2 > 0$  assures  $\hat{y} < \infty$ , i.e.  $\hat{y}$  is finite.

Solving the Pearson equation (B9)

$$(B9) \quad 0 = \left(\frac{\pi^2}{12}\right)\sigma^2m^2[y(t)]^2 \frac{\partial p_y(y_t, t)}{\partial y} - \left\{ b - \left[ (n + a - b + \left(\frac{\pi^2}{6}\right)\sigma^2m^2) \right] y(t) \right\} p_y(y_t, t)$$

$$\frac{\partial p_y(y_t, t)}{\partial y} = \left\{ \frac{b}{\left(\frac{\pi^2}{12}\right)\sigma^2m^2[y(t)]^2} - \frac{\left[ n + a - b + \left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right]}{\left(\frac{\pi^2}{12}\right)\sigma^2m^2 y(t)} \right\} p_y(y_t, t)$$

Hence

$$(B13) \quad \frac{\partial p_y(y_t, t)}{p_y(y_t, t)} = \left\{ \frac{b}{\left(\frac{\pi^2}{12}\right)\sigma^2m^2[y(t)]^2} - \frac{\left[ (n+a-b+\left(\frac{\pi^2}{6}\right)\sigma^2m^2) \right]}{\left(\frac{\pi^2}{12}\right)\sigma^2m^2 y(t)} \right\} \partial y$$

Solving the equation (B13) yields

$$(B14) \quad p_y(y_t, t) = ky \left( \frac{-12 \left[ n+a-b+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right]}{\pi^2\sigma^2m^2} \right) e^{\left\{ \frac{-12b}{\pi^2\sigma^2m^2 y} \right\}}$$

Where  $k$  is the integration coefficient and  $0 \leq y < \infty$

To find  $k$  I use the normalization requirement

$$\int_0^{\infty} p_y(y_t, t) dy = 1$$

Thus

$$\int_0^{\infty} ky \left\{ \frac{-12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} e^{\left\{ \frac{-12b}{\pi^2 \sigma^2 m^2 y} \right\}} dy = 1$$

Solving the integral yields

$$k^{-1} = \left( \frac{1}{q} \right)^{Q+1} \int_0^{\infty} T^Q e^{-T} dT = \left( \frac{1}{q} \right)^{Q+1} \Gamma(Q+1)$$

Where  $\Gamma$  is the gamma function and

$$q = \frac{12b}{\pi^2 \sigma^2 m^2}$$

$$T = \frac{12b}{\pi^2 \sigma^2 m^2 y}$$

$$Q = \left\{ \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} - 2$$

Hence

$$(B15) \quad k^{-1} = \left( \frac{12b}{\pi^2 \sigma^2 m^2} \right)^{\left\{ 1 - \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\}} \Gamma \left\{ \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} - 1 \right\}$$

To summaries, the overall solution of the Pearson equation (B9) is

$$(B14) \quad p_y(y_t, t) = ky \left\{ \frac{-12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} e^{\left\{ \frac{-12b}{\pi^2 \sigma^2 m^2 y} \right\}}$$

where

$$(B15) \quad k^{-1} = \left( \frac{12b}{\pi^2 \sigma^2 m^2} \right) \left\{ 1 - \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} \Gamma \left\{ \frac{12 \left[ n+a-b + \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} - 1 \right\}$$

I have defined above

$$(24) \quad y(t) = \frac{z(t)}{1-z(t)}$$

And found

$$(B4) \quad dz(t) = dy(t)[1 - z(t)]^2$$

Hence

$$\frac{dy(t)}{dz(t)} = \frac{1}{[1 - z(t)]^2}$$

Since  $y(t)$  is monotone increasing function with  $0 \leq y < \infty$ , thus  $z(t)$  is also monotone increasing function in  $y(t)$  in the domain of  $y(t)$  and  $0 \leq z < 1$ .

Since  $y(t)$  has a stationary distribution  $p_y(y_t, t)$ ,  $z(t)$  also have a stationary distribution  $p_z(z_t, t)$  such that

$$(B16) \quad p_z(z_t, t) = p_y(y_t, t) \left[ \frac{dy(t)}{dz(t)} \right] = p_y \left( y_t = \frac{z_t}{1-z_t} \right) \left\{ \frac{1}{[1-z(t)]^2} \right\}$$

Insert equation (B14) and equation (24) into (B16) to find

$$(B17) \quad p_z(z_t, t) = k e^{\left\{ \frac{12b}{\pi^2 \sigma^2 m^2} \right\}} (1-z) \left[ \frac{12(b+a-b)}{\pi^2 \sigma^2 m^2} \right]_z \left\{ \frac{12 \left[ b-a-n - \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 \right]}{\pi^2 \sigma^2 m^2} \right\} e^{\left\{ \frac{-12b}{\pi^2 \sigma^2 m^2 z} \right\}}$$

Where  $0 \leq z < 1$

$\hat{z}$  is the mode of a variable with stationary distribution thus satisfy  $\frac{\partial p_z(\hat{z}, t)}{\partial \hat{z}} = 0$

Insert equation (B17) into the above condition

$$\begin{aligned}
0 &= \frac{\partial p_z(\hat{z}, t)}{\partial \hat{z}} \\
&= ke^{\left\{\frac{12b}{\pi^2\sigma^2m^2}\right\}} \left\{ \left[ \frac{-12(b+a-b)}{\pi^2\sigma^2m^2} \right] (1-z)^{\left[\frac{12(b+a-b)}{\pi^2\sigma^2m^2}\right]-1} z^{\left\{\frac{12\left[b-a-n-\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]}{\pi^2\sigma^2m^2}\right\}} \right. \\
&\quad + (1-z)^{\left[\frac{12(b+a-b)}{\pi^2\sigma^2m^2}\right]} \left\{ \frac{12\left[b-a-n-\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]}{\pi^2\sigma^2m^2} \right\} z^{\left\{\frac{12\left[b-a-n-\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]}{\pi^2\sigma^2m^2}\right\}-1} \left. \right\} e^{\left\{\frac{-12b}{\pi^2\sigma^2m^2z}\right\}} \\
&\quad + e^{\left\{\frac{-12b}{\pi^2\sigma^2m^2z}\right\}} \left[ \frac{-12b}{\pi^2\sigma^2m^2} \right] (-1)z^{-2}(1-z)^{\left[\frac{12(b+a-b)}{\pi^2\sigma^2m^2}\right]} z^{\left\{\frac{12\left[b-a-n-\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]}{\pi^2\sigma^2m^2}\right\}} \left. \right\}
\end{aligned}$$

Solving for  $\hat{z}$  gives a quadratic equation

$$(B18) \quad \left(\frac{\pi^2}{6}\right)\sigma^2m^2\hat{z}^2 - \left[n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]\hat{z} + b = 0$$

Usually a quadratic equation has two solution. However, since  $0 \leq \hat{z} < 1$

$$\text{For } \hat{z} = 0 : \left(\frac{\pi^2}{6}\right)\sigma^2m^2 0^2 - \left[n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right] 0 + b > 0$$

$$\text{For } \hat{z} = 1 : \left(\frac{\pi^2}{6}\right)\sigma^2m^2 1^2 - \left[n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right] 1 + b = b - a - n < 0$$

Hence in the domain  $0 \leq \hat{z} < 1$  there is only one solution (cross the zero line) for equation (B18) which is equal to

$$(B19) \quad \hat{z} = \frac{\left[n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right] - \sqrt{\left[n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2\right]^2 - 4\left(\frac{\pi^2}{6}\right)\sigma^2m^2b}}{2\left(\frac{\pi^2}{6}\right)\sigma^2m^2}$$

### **Appendix C: deriving the unconditional mean and variance of the consumption growth**

I will derive the unconditional mean of consumption growth  $\frac{E\left[\frac{dc(t)}{c(t)}\right]}{dt}$   
From appendix A

$$(A30) \quad \frac{dc(t)}{c(t)} = \left[ n + b - \frac{(n+a)x(t)}{c(t)} \right] dt + \left[ 1 - \frac{x(t)}{c(t)} \right] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t)$$

where

$$(A22) \quad n = \frac{(\mu-r+\varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu-r+\varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

I will calculate the average on both sides of equation (A30) and insert equation (22)

$$E \left[ \frac{dc(t)}{c(t)} \right] = \left[ n + b - (n+a)E[z(t)] \right] dt + \{1 - E[z(t)]\} \left( \frac{\pi}{\sqrt{6}} \right) \sigma m E[dw(t)]$$

Since for a wiener process  $E[w(t)] = 0$  thus

$$E[dw(t)] = d\{E[w(t)]\} = 0$$

Hence

$$E \left[ \frac{dc(t)}{c(t)} \right] = \{n + b - (n+a)E[z(t)]\} dt + 0$$

So

$$(C1) \quad \frac{E\left[\frac{dc(t)}{c(t)}\right]}{dt} = n + b - (n+a)E[z(t)] = n + b - (n+a) \int_0^1 z(t) p_z(z_t, t) dz$$

Since a closed form expression for the integral is unavailable, the integration is done numerically.

I will calculate the variance on both sides of equation (A30) and insert equation (22)

$$\begin{aligned} var \left[ \frac{dc(t)}{c(t)} \right] &= var\{[n + b - (n+a)z(t)]dt\} + var \left\{ [1 - z(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t) \right\} + \\ &+ 2cov \left\{ [n + b - (n+a)z(t)]dt, [1 - z(t)] \left( \frac{\pi}{\sqrt{6}} \right) \sigma m dw(t) \right\} = \end{aligned}$$

Since  $var(dt) = 0$ ,

And for a wiener process  $var[dw(t)] = dvar[w(t)] = dt$  I derive

$$var \left[ \frac{dc(t)}{c(t)} \right] = \left( \frac{\pi^2}{6} \right) \sigma^2 m^2 E\{[1 - z(t)]^2\} dt$$

hence

$$(C2) \quad \frac{\text{var}\left[\frac{dc(t)}{c(t)}\right]}{dt} = \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 E\{[1 - z(t)]^2\} = \left(\frac{\pi^2}{6}\right) \sigma^2 m^2 \int_0^1 [1 - z(t)]^2 p_z(z_t, t) dz$$

Since a closed form expression for the integral is unavailable, the integration is done numerically.

### **Appendix D: deriving RRA coefficient and the intertemporal Elasticity of Substitution in Consumption (=s)**

In appendix A I derived

$$(A10) \quad V[W(t), x(t)] = \frac{(h)^\gamma}{\gamma} \left[ W(t) - \frac{x(t)}{r+a-b} \right]^\gamma \left( \frac{-1}{H} \right)$$

The derivatives of equation (A10)

$$V_W = \frac{(h)^\gamma}{\gamma} \gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right) = (h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right)$$

$$V_{WW} = (h)^\gamma (\gamma - 1) \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-2} \left( \frac{-1}{H} \right)$$

I define RRA coefficient and insert the above derivatives

$$(D1) \quad RRA = \frac{-WV_{WW}}{V_W} = \frac{-W(h)^\gamma (\gamma - 1) \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-2} \left( \frac{-1}{H} \right)}{(h)^\gamma \left[ W(t) - \frac{x(t)}{r+a-b} \right]^{\gamma-1} \left( \frac{-1}{H} \right)} = (1 - \gamma) \frac{1}{1 - \frac{x(t)}{W(r+a-b)}}$$

From equation (A3)

$$(A3) \quad c(t) - x(t) = h \left[ W(t) - \frac{x(t)}{r+a-b} \right]$$

Solving for  $W(t)$

$$(D2) \quad W(t) = \left[ \frac{c(t) - x(t)}{h} \right] + \frac{x(t)}{r+a-b}$$

Insert equation (D2) into (D1)



$$RRA = (1 - \gamma) \left\{ \frac{1}{1 - \frac{x(t)}{\left[ \frac{c(t) - x(t)}{h} \right] + \frac{x(t)}{r + a - b}} (r + a - b)} \right\} =$$

$$= (1 - \gamma) \left\{ 1 + \left[ \frac{x(t)}{c(t) - x(t)} \right] \left[ \frac{h}{(r + a - b)} \right] \right\}$$

Inserting equation (24) into the above expression

$$(24) \quad y(t) = \frac{x(t)}{c(t) - x(t)} = \frac{z(t)}{1 - z(t)}$$

thus

$$(D3) \quad RRA = (1 - \gamma) \left\{ 1 + y(t) \left[ \frac{h}{(r + a - b)} \right] \right\}$$

From equation (D3) and since  $y(t)$  has a steady state distribution (with  $\bar{y}$  and  $\hat{y}$ ), RRA coefficient also has a steady state distribution. Thus RRA average is

$$\overline{RRA} = (1 - \gamma) \left\{ 1 + \overline{y(t)} \left[ \frac{h}{(r + a - b)} \right] \right\}$$

Insert equation (28) into the above equation

$$(28) \text{ (B11)} \quad \bar{y} = \frac{b}{n + a - b}$$

Thus

$$(D4) \quad \overline{RRA} = (1 - \gamma) \left\{ 1 + \left[ \frac{b}{n + a - b} \right] \left[ \frac{h}{(r + a - b)} \right] \right\} = (1 - \gamma) \left\{ 1 + \left[ \frac{hb}{(n + a - b)(r + a - b)} \right] \right\}$$

When  $z(t) = \hat{z}$

$$(D5) \quad RRA(z = \hat{z}) = (1 - \gamma) \left\{ 1 + \left[ \frac{\hat{z}}{1 - \hat{z}} \right] \left[ \frac{h}{(r + a - b)} \right] \right\}$$

The elasticity of substitution in consumption (=s) is defined as the derivative of the expected growth rate in consumption with respect to  $r$  while  $z(t)$ ,  $\mu - r$ , and  $\sigma^2$  held constant thus

$$s = \frac{\partial \left[ \frac{E\left(\frac{dc}{c}\right)}{dt} \right]}{\partial r} \quad \text{while } z(t), \mu - r, \text{ and } \sigma^2 \text{ held constant}$$

From equation (C1) when  $z(t), \mu - r, \text{ and } \sigma^2$  held constant

(C1)

$$\frac{E \left[ \frac{dc(t)}{c(t)} \right]}{dt} = n + b - (n + a)E[z(t)] = n + b - (n + a)z(t) = n[1 - z(t)] + b - az(t)$$

And equation (20)

$$(20) \text{ (A22)} \quad n = \frac{(\mu - r + \varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho - \gamma r + \frac{(\mu - r + \varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\} =$$

$$= r \left( \frac{1}{1-\gamma} \right) + \frac{(\mu - r + \varepsilon\sigma)^2}{\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} + r - \frac{1}{(1-\gamma)} \left\{ \rho + \frac{(\mu - r + \varepsilon\sigma)^2}{4\left(\frac{\pi^2}{6}\right)\sigma^2(1-\gamma)} \right\}$$

Insert equation (20) into (C1) and derive  $s$  when  $z(t), \mu - r, \text{ and } \sigma^2$  held constant

$$(D6) \quad s = \frac{\partial \left[ \frac{E\left(\frac{dc}{c}\right)}{dt} \right]}{\partial r} = \left[ 1 \left( \frac{1}{1-\gamma} \right) + 0 - 0 \right] [1 - z(t)] + 0 = \frac{1-z(t)}{1-\gamma}$$

To find  $s^*RRA$  use equations (D6), (24) and (D3)

$$(D7) \quad s * RRA = \left[ \frac{1-z(t)}{1-\gamma} \right] (1-\gamma) \left\{ 1 + y(t) \left[ \frac{h}{(r+a-b)} \right] \right\} =$$

$$= [1 - z(t)] \left\{ 1 + \left[ \frac{z(t)}{1-z(t)} \right] \left[ \frac{h}{(r+a-b)} \right] \right\} = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} z(t)$$

Hence

$$(D7) \quad s * RRA = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} z(t)$$

The modal value of s\*RRA is

$$(D8) \quad mode(s * RRA) = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} \hat{z}$$

Thus for a time separable utility (i.e. b=0) the following values found  
Using equations (B19) and (D8)

$$(B19) \quad \hat{z} = \frac{\left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right] - \sqrt{\left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right]^2 - 4\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \cdot 0}}{2\left(\frac{\pi^2}{6}\right)\sigma^2m^2} = \frac{\left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right] - \left[ n+a+\left(\frac{\pi^2}{6}\right)\sigma^2m^2 \right]}{2\left(\frac{\pi^2}{6}\right)\sigma^2m^2} = 0$$

$$(D8) \quad mode(s * RRA) = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} \hat{z} = 1 - \left\{ 1 - \left[ \frac{h}{(r+a-b)} \right] \right\} 0 = 1$$

### **Appendix E: derive mean and variance of risky asset return**

Using equation (47)

$$(47) \quad \frac{dS(t)}{S(t)} = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right] dt + \left( \frac{\delta_1}{\delta_2} \right) \left( \frac{\pi}{\sqrt{6}} \right) \sigma dw(t)$$

Apply expectation operator on both sides

$$E \left[ \frac{dS(t)}{S(t)} \right] = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right] E[dt] + \left( \frac{\delta_1}{\delta_2} \right) \left( \frac{\pi}{\sqrt{6}} \right) \sigma E[dw(t)]$$

Since  $E[dt] = dE[t] = dt$  and

$E[dw(t)] = dE[w(t)] = 0$  since for a wiener process  $E[w(t)] = 0$  I derive

$$E \left[ \frac{dS(t)}{S(t)} \right] = \left[ \frac{\delta_1}{\delta_2} (\mu + \varepsilon\sigma - r) + r \right] dt$$

Hence

$$(E1) \quad \frac{E\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left[\frac{\delta_1}{\delta_2}(\mu + \varepsilon\sigma - r) + r\right]$$

Apply variance on both sides of equation (44) above

$$\begin{aligned} \text{var}\left[\frac{dS(t)}{S(t)}\right] &= \left[\frac{\delta_1}{\delta_2}(\mu + \varepsilon\sigma - r) + r\right]^2 \text{var}[dt] + \left[\left(\frac{\delta_1}{\delta_2}\right)\left(\frac{\pi}{\sqrt{6}}\right)\sigma\right]^2 \text{var}[dw(t)] + \\ &+ 2\text{cov}\left\{\left[\frac{\delta_1}{\delta_2}(\mu + \varepsilon\sigma - r) + r\right]\left(\frac{\delta_1}{\delta_2}\right)\left(\frac{\pi}{\sqrt{6}}\right)\sigma(dt)dw(t)\right\} \end{aligned}$$

since

$\text{var}[dt] = d\text{var}(t) = 0$  and for a wiener process  $(dt)dw(t) = 0$ ,  $\text{var}[w(t)] = t$  thus

$\text{var}[dw(t)] = d\text{var}[w(t)] = dt$  hence

$$\text{var}\left[\frac{dS(t)}{S(t)}\right] = \left[\left(\frac{\delta_1}{\delta_2}\right)\left(\frac{\pi}{\sqrt{6}}\right)\sigma\right]^2 dt$$

so

$$(E2) \quad \frac{\text{var}\left[\frac{dS(t)}{S(t)}\right]}{dt} = \left[\left(\frac{\delta_1}{\delta_2}\right)\left(\frac{\pi}{\sqrt{6}}\right)\sigma\right]^2 = \left(\frac{\delta_1}{\delta_2}\right)^2 \left(\frac{\pi^2}{6}\right)\sigma^2$$

## References

- AKDENIZ, L. and W.D. DECHERT (2007), "The Equity Premium in Brock's Asset Pricing Model," *Journal of Economic Dynamics and Control*, vol. 31, pp. 2263-2292
- BROCK, W. A. (1982), "Asset Prices in a Production Economy," in *The Economics of Information and Uncertainty*, ed. by J. J. McCall, pp.1-46. The University of Chicago Press, Chicago.
- CAMPBELL, J.Y. and COCHRANE, J.H. (1999), "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", *The Journal of Political Economy*, Volume 107, Issue 2, 205-251.
- CONSTANTINIDES, G. M. (1990), "[Habit Formation: A resolution of the Equity Premium Puzzle](#)," *Journal of Political Economy* 98 (3): 519-543.
- DAVIS, M. H. A. and NORMAN, A. R. (1987), "Portfolio selection with Transaction Costs." Manuscript. London: Imperial Coll.
- DUESENBERY, J.S. (1949), "Income, Saving and the Theory of Consumer Behavior", *Harvard University Press*, Cambridge.
- DUNN, K. B., and SINGLETON, K. J. (1986), "Modeling the Term Structure of Interest Rates under Non-separable Utility and Durability of Goods." *J. Financial Econ.* 17 (September 1986): 27-55.
- EICHENBAUM, M. S., and HANSEN, L. P. (1990). "Estimating Models with Intertemporal Substitution Using Aggregate Time Series Data." *J. Bus. and Econ. Statis.*
- EICHENBAUM, M. S.; HANSEN, L. P. and SINGLETON, K. J. (1988), "A Time Series Analysis of Representative Agent Models of Consumption and Leisure Choice under Uncertainty." *Q.J.E.* 103 (February 1988): 51-78.
- FERSON, W. E. (1983), "Expectations of Real Interest Rates and Aggregate Consumption: Empirical Tests." *J. Financial and Quantitative Analysis* 18 (December 1983): 477-97.
- FERSON, W. E. and CONSTANTINIDES, G. M. (1989), "Habit Persistence and Durability in Aggregate Consumption: Empirical Tests." Manuscript. Chicago: Univ. Chicago.
- FRIEND, I. and BLUME, M.E. (1975), "The demand for risky assets", *American Economic Review* 65, 900-922.

- GALLANT, A. R. and TAUCHEN, G. (1989), "Seminonparametric Estimation of Conditionally Constrained Heterogeneous Processes: Asset Pricing Applications." Manuscript. Durham, N.C.: Duke Univ.
- GROSSMAN, S. J.; MELINO, A.; and SHILLER, R. J. (1987), "Estimating the Continuous-Time Consumption-based Asset-Pricing Model." *J. Bus. And Econ. Statis.* 5 (July 1987): 315-27.
- HANSEN, L. P. and JAGANNATHAN, R. (1988), "Restrictions on Intertemporal Marginal Rates of Substitution Implied by Asset Prices." Manuscript. Chicago: Univ. Chicago, 1988.
- HANSEN, L. P. and SINGLETON, K. J. (1982), "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models." *Econometrica* 50 (September 1982): 1269-86.
- HANSEN, L. P. and SINGLETON, K. J. (1983), "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns." *J. P. E.* 91 (April 1983): 249-65.
- HEATON, J. (1988), "The Interaction between Time-Nonseparable Preferences and Time Aggregation." Manuscript. Chicago: Univ. Chicago.
- IBBOTSON, R. G. and SINQUEFIELD, R. A. (1982), "Stocks, Bonds, Bills and Inflation: The Past and the Future. Charlottesville, Va.: *Financial Analysts Res. Found.*
- KOCHERLAKOTA, N. R. (1996), "[The Equity Premium: It's Still a Puzzle](#)," *Journal of Economic Literature* 34 (1): 42-71.
- LUCAS, R. E. (1978), "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.
- MEHRA, R., and E.C. PRESCOTT (1985), "[The Equity Premium: A Puzzle](#)". *Journal of Monetary Economics* 15: 145-161.
- MCGRATTAN, E. R., and E.C. PRESCOTT (2001), "Taxes, Regulations, and asset Prices". Working paper 610, Federal Reserve Bank of Minneapolis.
- RIETZ, T. A. (1988), "The Equity Risk Premium: A Solution." *J.Monetary Econ.* 22(July 1988): 117-31.
- SHILLER, R. J, (1981), "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review*, 71 (3): 421-436.