Misallocation and Growth

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Abstract

Endogenous growth model in which training is a by-product of production, and when both firms and workers are heterogeneous. Two-period OLG. The young must work with the old, and as a result an assignment problem arises. We can calculate the transition dynamics analytically when the skill of the young is log-normally distributed and the initial human capital of the old generation is also log-normal. Assignment frictions translate only into level effects, with no effect on long-run growth. But rates of convergence to the BGP are slow. When types are private information, a sepaating equilibrium nonetheless emerges in which the allocations stay the same as in the full-information case, but not the prices PRELIMINARY – DO NOT CIRCULATE

1 Introduction

Work on misallocation tends to be mostly about levels of activity rather than growth, and highlights frictions of search (Burdett and Coles 1998, Shimer and Smith 2000, Lagos 2006, Anderson and Smith 2010) or implicit taxes (Restucchia and Rogerson 2008, Hsieh & Klenow 2008). Papers on growth highlight the misallocation of intermediate goods (Jones 2011) and of capital due to finance (Greenwood and Jovanovic 1990, Buera and Shin 2009).

This paper is about labor-market misallocation and growth. Growth arises because of on-the-job training and because of external effects flowing from the old to the young via some process such as the quality of elementary schooling or upbringing by parents. Both mechanisms are in Lucas (1988) in a representative agent model. Boyd & Prescott (1987) is similar because it is two-period lived overlapping generations, because there is no vintage capital and because growth is endogenous. In the model firms are heterogeneous, and the skills that a young agent can learn differ

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from firm to firm as in Chari & Hopenhayn (1991), Jovanovic and Nyarko (1995) and Monge-Naranjo (2011) and wages include a compensating differential.[1] Even within these models one can study misallocation of labor across firms or technologies and it would have consequences for growth in the Boyd-Prescott model, and for the transitional dynamics in the Chari-Hopenhayn model. My model adds heterogeneity to the abilities of each young generation so that a Becker-Brock type of assignment problem arises (Becker 1973). Firms are perfect competitors in both the product and factor markets.

With no frictions in how workers are assigned to firms, the model is of an "Ak" type, with no transitional dynamics. With heterogeneity among the talent of the young, transitional dynamics arise, and in the long run, the inequality among the old is determined by the exogenous inequality at birth in innate skills. We then add a friction to allocation in the factor market. The friction is expressed in the noise in a public signal of a worker's ability; the noise impedes the market's ability to reach a positive assignment, an idea that is in MacDonald (1982) and Kremer (1993).[2] The friction turns out to affect long-run inequality of incomes, but not the long-run growth rate. Moreover, when the talent of the young is log-normally distributed, we can calculate the dynamics in closed form when the initial distribution of human capital among the old (which also is the state of the system) is itself log normal.

Questions that we can answer here are: At the macro level, how large a development gap can such a friction explain? A figure towards the end of the paper answers this question visually; a garbled assignment can be damaging for long-run development, but based on some estimates that Heathcote *et al.* (2005) report, the development gap explained by the model is not likely to exceed 10 percent of the leader. The question is answered also for the transitional episode that may itself be triggered by a change in the accuracy with which skills are measured.[3] At the micro level we find that output and inequality rise when the garbling frictions diminish and assignments more accurately reflect workers' talents.

The model is developed under the assumption that complementarity in production and training exists so that the equilibrium will lead to monotone positive sorting between bosses and workers. A parallel analysis could be developed for monotone negative sorting. Things would rapidly become messy, however, if the production functions did not have a cross partial that changes sign on its domain.

[1] There is a large literature beginnning with Nelson and Phelps (1968) in which spillovers of knowledge occur through channels that do not entail payments; papers that focus on this are Jovanovic and Rob (1989), Benhabib and Spiegel (1994), Jovanovic and MacDonald (1994), Barro and Sala-i-Martin (1997), Eeckhout and Jovanovic (2002), Perla and Tonetti (2011), and Lucas and Moll (2011).

[2] In MacDonald (1982), a continuum of imperfectly labeled types is assigned to two tasks. Kremer generalizes to a continuum types and tasks.

[3] Ljungqvist and Sargent (1998) and Violante (2002) also study the evolution of inequality in response to a change of regime.

Evidence

A. *Substitutability in production and training*: Let us list some evidence on this assumption.

(*i*) *Skilled-Unskilled.*—Griliches (1989), Krusell *et al.* (2000). Katz and Murphy (1992) elasticity of substitution is 1.4.

(*ii*) *Boss-worker.*—Lazear, Shaw, and Stanton (2011) find a complementarity between the qualities of bosses and their workers. Andersson *et al.* (2006) find strong relationships between a firm's existing talent, newly hired talent, and overall quality of software firms. Gabaix and Landier (2008) find that better CEOs match with larger firms. Gavilan (2012) documents some recent rise in segregation of worker quality in plants in the U.S. presumably because qualities became more complementary.

(*iii*) *Teacher-student.*—Hanushek (2011) and the empirical literature on economics of education usually assumes a cross-partial derivative that is zero, which allows the use of linear fixed-effect models. An exception is Lockwood and McCaffrey (2009) who find weak positive complementarity between quality of teachers and students. Behrman, Todd and Wolpin (in progress) find complementarity between teacher and student's efforts in generating student achievement.

(*iv*) *Parent-child.*—Cunha, Heckman and Schennach (2010) estimate a CES function of the form $\mathrm{H}_{t+1} = \mathrm{F}(\mathrm{H}_t,\mathrm{I}_t,\text{Parental H})$. The elasticity of substitution is around 1.2 when the child is 0-4 yrs old and 0.6 when the child is older than 5yrs. In other words, complementarity between parent and child quality grows with age).

B. *Experience premium.*—2-5 percent per year. Gottschalk (1997), Jeong, Kim and Manovski (2011).

C. *Gibrat's Law*, etc.— Luttmer (2010), Sutton (1997).

D. *Misallocation.*—And finance, Midrigan and Xu (2010), and occupational choice Hsieh *et al.* (2012).

Plan of paper.—Section 2 deals with homogeneous generations; its purpose is to clarify the growth mechanism. Section 3 adds heterogeneity in each young generation and solves for the balanced growth path prices and allocations. Section 4 introduces a particular friction that leads to a garbled assignment and that features transition dynamics.

2 Homogeneous generations

The model is a 2-period overlapping generations with lifetime utility depending linearly on consumption in youth, c_{y}, and consumption in old age, c_{o}:

$$c_{\mathrm{y}} + \beta c_{\mathrm{o}}$$

An agent is endowed with a unit of labor in both periods of life. Population is fixed, with a unit measure of young being born each period. Let x_t denote the skill of each old agent and y_t the skill of each young agent at date t

Firms.—A firm must have one old & one young worker. The idea is that a manager must be more experienced than the workers he oversees. The firm produces a consumption good and training using the following production functions:

$$\begin{aligned} \text{output} \qquad q_t &= f(x_t, y_t) && (1) \\ \text{training} \qquad x_{t+1} &= \phi(x_t, y_t) && (2) \end{aligned}$$

where f and ϕ are homogeneous of degree one. Boyan's current model is a natural progression from his 1995 paper with Nyarko in which the old transfer information to the young. There an asymmetric information problem prevented efficiency, while in the current work the welfare theorems obtain.[4]

External effect.—Let ε denote as an innate talent parameter for a young agent. In this section ε is the same for each young agent in each generation. There is an externality

$$y_t = \psi(\bar{x}_t, \varepsilon) \tag{3}$$

The term $\bar{x}_t$ is the skill of the each old agent, and it exerts a positive spillover effect on the marketable skill of the young. Since the old agents are all the same,

$$\bar{x}_t = x_t.$$

Substituting from (3) into (1) we find that output is

$$q_t = f(x_t, \psi(x_t, \varepsilon)) \tag{4}$$

LR growth.—When is LR growth feasible, at least from some initial conditions? Let

$$\psi(x, \varepsilon) = b\varepsilon x > 0. \tag{5}$$

[4] A version of (1) and (2) is analyzed by Andersen and Smith (2010) and Andersen (2011); they allow for randomness in (2), but they specialize in other ways: They analyze a partnership model with no defined roles and symmetric production and transitions, and their dynamic model degenerates into a static model whenever perfect sorting obtains, since in that case no types change. Their focus has thus been characterizing matching patterns that are not perfectly assortative, and the resulting type, wage, and inequality dynamics. In my model, by contrast, the old and young have defined roles in production, different distributions over human capital, and interesting dynamics obtain even with assortative matching. So these two papers on the one hand and mine on the other have simplified along different dimensions toward a different sort of theory, and studying each of these cases in isolation appears to simplify the task.

Then

$$\frac{x_{t+1}}{x_t} = \phi\left(1, \frac{1}{x_t}\psi\left(x_t, \varepsilon\right)\right) = \phi\left(1, b\varepsilon\right) \equiv \Gamma. \tag{6}$$

Output then also grows at the same rate:

$$\frac{q_{t+1}}{q_t} = \frac{f\left(x_{t+1}, \psi\left(x_{t+1}, \varepsilon\right)\right)}{f\left(x_t, \psi\left(x_t, \varepsilon\right)\right)} = \frac{x_{t+1}}{x_t}\frac{f\left(1, \frac{1}{x_{t+1}}\psi\left(x_{t+1}, \varepsilon\right)\right)}{f\left(1, \frac{1}{x_t}\psi\left(x_t, \varepsilon\right)\right)} = \frac{x_{t+1}}{x_t}\frac{f\left(1, b\varepsilon\right)}{f\left(1, b\varepsilon\right)} = \Gamma.$$

To sum up: A CRS model (1)-(2) with a fixed factor y is converted into an Ak model via the external effect (3) making y a *de-facto* reproducible factor. The homogeneous-generation model has no transitional dynamics.

Example.—Let ϕ be Cobb-Douglas:

$$\phi\left(x, y\right) = Ax^{1-\theta}y^{\theta} \tag{7}$$

Then (6) implies that

$$\Gamma = A\left(b\varepsilon\right)^{\theta}. \tag{8}$$

We add heterogeneity next. We shall now get some transitional dynamics, but long-run growth will, essentially, be the same as that in (6).

3 Heterogeneous young with no frictions

We continue to maintain the production function f in (1) and the training function ϕ in (2). But the inputs (x, y) in both will now be heterogeneous. The state of the system will be the C.D.F. of x denoting the distribution of the skills among the members of the current old generation. Denoted by $H\left(x\right)$, this distribution will also summarize a direct influence that the skills of the old will exert on the skills of the young. This influence will be external, perhaps operating through the schooling mechanism. The C.D.F. of skills among the young will be denoted by G. To summarize,

$$\begin{aligned} x &= \text{skill of an old agent } x \sim H\left(x\right), \\ y &= \text{skill of a young agent } y \sim G^{\mathrm{H}}\left(y\right), \end{aligned}$$

where H represents external effect.

Why the external effect in G?—Why do we assume an external effect of the type $y_t = \psi\left(\bar{x}_t, \varepsilon\right)$ as displayed in eq. (3)? This is because the young are heterogeneous, and because the assignment of young to old is one-to-one. In particular, we certainly do not need the external effect (3) if the young are homogeneous, as indeed they were in section 2. If y were the same for all the young, we could replace (1) and (2) by

$$q = Ax\hat{f}(y) \quad \text{and} \quad x' = x\hat{\phi}(y), \tag{9}$$

we could drop (3), and x and q would then grow at the constant rate $\hat{\phi}(y)$. But if there is more than one type y and if $\hat{\phi}$ is increasing in y so that the equilibrium assignment of y to x is positive at each date, then (9) would yield a permanently higher growth for firms x that employ the high y type, and the coefficient of variation of x would go to infinity, contrary to what the data seem to show. Income inequality does not appear to be exploding. To prevent that from happening, we introduce diminishing returns to both x and y, but then generate growth in y via a linear external effect of x_{H}, and this allows both growth and inequality to settle down on the BGP.

3.1 Recursive equilibrium

In all that follows, the functions f and ϕ will be assumed to possess positive cross partials and that a positive assignment of y to x obtains at each date. We shall describe a Markovian equilibrium in which the valuation, $\pi(x)$, of human capital x by an old agent is defined recursively. Wages will then be stated in terms of $\pi(.)$. Carrying the state "H" as an argument will go on for a few equations, and we shall drop it after equilibrium is stated.

Assignment.—The assignment of the young to the old will be written as $y = \alpha(x)$. For a positive-sorting assignment α to clear the market requires that

$$H(x) = G^{\mathrm{H}}(y) \tag{10}$$

for all x, so that the assignment is

$$\alpha^{\mathrm{H}}(x) = \left(G^{\mathrm{H}}\right)^{-1}(H[x]). \tag{11}$$

Aggregate law of motion χ.—This function maps the set of C.D.F.s on the positive line into itself. Letting a prime "′" denote the next-period's value of H,

$$H' = \chi(H), \tag{12}$$

Let $x' = \xi^{\mathrm{H}}(x)$ denote the evolution of productivity in firm x when the aggregate state is H. That is,

$$\xi^{\mathrm{H}}(x) = \phi\left(x, \alpha^{\mathrm{H}}(x)\right) \tag{13}$$

Adding up we require that for all x', the number of tomorrow's old with skills $\leq x'$ be the same as the number of today's old x that, when matched with a young person of type $\alpha(x)$, will generate training $\leq x'$. I.e., for all x', H' must satisfy $H'(x') = H\left[\left(\xi^{\mathrm{H}}\right)^{-1}(x')\right]$. Stated compactly, the aggregate law of motion is

$$\chi(H) = H \cdot \left(\xi^{\mathrm{H}}\right)^{-1}. \tag{14}$$

It represents an equilibrium evolution if we can show that the positive assignment – which has so far been assumed to hold at each date – can be decentralized. Note, however, that (11) and (14) do not refer to prices which means that the calculation of the equilibrium is itself recursive – the quantities can be calculated first, and then the prices.

Wages and profits.—Denote by $w^{\mathrm{H}}(x)$ the wage paid by firm x in state H. The profits of firm x and the consumption of an old agent of type x then are

$$\pi^{\mathrm{H}}(x) = f\left(x, \alpha^{\mathrm{H}}(x)\right) - w^{\mathrm{H}}(x) \tag{15}$$

Equilibrium lifetime utility of a young agent of type y in state H is:

$$V^{\mathrm{H}}(y) = w^{\mathrm{H}}\left(\left(\alpha^{\mathrm{H}}\right)^{-1}(y)\right) + \beta\pi^{\chi(\mathrm{H})}\left(\phi\left[\left(\alpha^{\mathrm{H}}\right)^{-1}(y), y\right]\right) \tag{16}$$

It depends on wages received in youth, and on the training carried into old age.[5]

Firm x's decision problem.—Firm x chooses a type to match with, y, and a wage to pay him, w, so as to maximize its consumption and subject to providing agent y with his equilibrium lifetime utility

$$\pi^{\mathrm{H}}(x) = \max_{w,y}\{f(x,y) - w\} \tag{17}$$

s.t.

$$w + \beta\pi^{\chi(\mathrm{H})}(\phi[x,y]) \geq V^{\mathrm{H}}(y). \tag{18}$$

Now, (18) must bind at an optimum, and it can be used to eliminate w from (17) to yield a Bellman equation for π:

$$\pi^{\mathrm{H}}(x) = \max_{y}\left\{f(x,y) - V(y) + \beta\pi^{\chi(\mathrm{H})}(\phi[x,y])\right\}. \tag{19}$$

This equation takes χ as parametrically given, and χ is given in (14) with no reference to π, w, or V. This is because quantities can be calculated with no reference to prices.

Definition of equilibrium.—It consists of functions α and χ satisfying (11) and (14), and π, w, and V satisfying (16), (17), and (19) with

$$\alpha^{\mathrm{H}}(x) = \arg\max_{y}\{\text{RHS of (19)}\}. \tag{20}$$

[5]Note the updating of H to $\psi(H)$ on the RHS of (16).

4 The BGP under full information

To have a balanced-growth path (BGP) on which long-run growth in x coexist with a CRS form for ϕ in (2), we shall need the spillover that is captured by the effect of H in $G^{\mathrm{H}}(y)$ to roughly raise the talent of the young to a level proportional to the talent of the old.

4.0.1 Spillover mechanism

Let $\bar{x}_{\mathrm{H}} = \int x dH(x)$. A young agent with raw talent ε will then have market skill.

$$y = b\bar{x}_{\mathrm{H}}\varepsilon. \tag{21}$$

In other words, except for the substitution of $\bar{x}_{\mathrm{H}}$ for $\bar{x}_t$, (21) is the same as (5) in how a given agent's ε translates into his y. But now the young differ in their raw talents: ε is assumed to be distributed according to the C.D.F. $\hat{G}(\varepsilon)$, assumed to be identical over generations. Together with (21), this implies that y is distributed according to the C.D.F.

$$G^{\mathrm{H}}(y) = \hat{G}\left(\frac{y}{b\bar{x}_{\mathrm{H}}}\right). \tag{22}$$

Now we explain how even under heterogeneity, the mechanism in (21) is likely to enable a balanced-growth path (BGP) to exist. the role that For any C.D.F. $H(x)$ and constant $\Gamma > 0$, let the resulting distribution of $\tilde{x} = \Gamma x$ be denoted by

$$H^{\Gamma}(\tilde{x}) = H\left(\frac{\tilde{x}}{\Gamma}\right). \tag{23}$$

The assignment that enables H to evolve into H^{Γ} must be homogeneous of degree one in (x, Γ):

Proposition 1 *If (21) holds,*

$$\alpha^{H^{\Gamma}}(x) = \Gamma\alpha^{H}\left(\frac{x}{\Gamma}\right) \tag{24}$$

Proof. $\alpha^{\mathrm{H}^{\Gamma}}(x)$ solves for y the equation (10), which reads

$$H^{\Gamma}(x) = G^{\mathrm{H}^{\Gamma}}(y). \tag{25}$$

But according to (22), $G^{\mathrm{H}^{\Gamma}}(y) = \hat{G}\left(\frac{y}{bx_{\mathrm{H}^{\Gamma}}}\right) = \hat{G}\left(\frac{y}{b\Gamma x_{\mathrm{H}}}\right) = G^{\mathrm{H}}\left(\frac{y}{\Gamma}\right)$, the second equality follows because $\bar{x}_{\mathrm{H}^{\Gamma}} = \Gamma\bar{x}_{\mathrm{H}}$. Substituting into (25) and using (23), $\alpha^{\mathrm{H}^{\Gamma}}(x)$ solves for y the equation

$$H\left(\frac{x}{\Gamma}\right) = G^{\mathrm{H}}\left(\frac{y}{\Gamma}\right) \Rightarrow \alpha^{\mathrm{H}^{\Gamma}}(x) = \Gamma\left(G^{\mathrm{H}}\right)^{-1}\left[H\left(\frac{x}{\Gamma}\right)\right] = \Gamma\alpha^{\mathrm{H}}\left(\frac{x}{\Gamma}\right)$$

in light of (11), i.e., (24). ■

Lemma 1 shows that if all agents' x's are multiplied by Γ, each old agent will be able to match a young agent with quality Γy, where y was that agent's previous assignment. In other words, regardless of H, if every x were to double, the quality of the equilibrium assignment of y's would double too.

The result in (24) does not invoke the linear homogeneity of ϕ; it relies only on (21) and on the assumption that each x has grown by the same proportion. But the result makes it clear that since f and ϕ indeed are both linearly homogeneous, the economy is capable of growing at a constant rate once a particular H is attained. If it reaches such an H, the economy will then be on its BGP.

4.0.2 (α, Γ, H) on the BGP

Recall that we can solve for the quantities first, and the prices later. So, we first solve for the triple (α, Γ, H) on the BGP.

Proposition 2 *On the BGP,*

$$\alpha(x) = b\bar{\varepsilon}x, \qquad \text{and} \tag{26}$$

$$H(x) = \hat{G}\left(\frac{\varepsilon}{C}\right), \tag{27}$$

where $C > 0$ is any real number. If, at $t = 0$, $H_0 = \hat{G}\left(\frac{\varepsilon}{C_0}\right)$, then

$$H_t = \hat{G}\left(\frac{\varepsilon}{C_0\Gamma^t}\right) \tag{28}$$

where

$$\Gamma = \phi(1, b\bar{\varepsilon}) \tag{29}$$

is the growth factor.

Proof. Suppose the economy is at H. If each x grows at some common factor Γ, in terms of (13), this means that for all x in the support of H, $\Gamma x = \phi\left(x, \alpha^{\mathrm{H}}(x)\right)$or, dividing both sides by x, that

$$\Gamma = \frac{1}{x}\phi\left(x, \alpha^{\mathrm{H}}(x)\right) = \phi\left(1, \frac{\alpha^{\mathrm{H}}(x)}{x}\right)$$

where the second equality follows by the linear homogeneity of ϕ. This means that

$$\alpha^{\mathrm{H}}(x) = \alpha^* x, \tag{30}$$

where $\alpha^* > 0$ is a constant satisfying. $\phi(1, \alpha^*) = \Gamma$. Evaluating the RHS of (30) at the means, $\bar{y} = \alpha^*\bar{x} = b\bar{x}\bar{\varepsilon}$ (using (21)), which, together with (30) implies (26). Dividing $\alpha^*\bar{x} = b\bar{x}\bar{\varepsilon}$ by $\bar{x}$ implies

$$\alpha^* = b\bar{\varepsilon}, \tag{31}$$

which proves (29). Next, (21) and (30) imply $\alpha^* x = b\bar{x}\varepsilon$, and this leads to

$$\frac{x}{\bar{x}} = \frac{b}{\alpha^*}\varepsilon = \frac{\varepsilon}{\bar{\varepsilon}}$$

where the second equality follows from (31). Therefore x is a scaled version of ε which proves (27). Finally, with a constant growth factor Γ, (28) follows ■

Remarks on Proposition 2:

1. Notice that (29) and (6) become the same if the ε's become identical.
2. Growth depends positively on the talent of the young and on the externality parameter b, as well as on the parameters of ϕ. E.g.,

   $$\phi(x, y) = Ax^{1-\theta}y^{\theta} \Rightarrow \Gamma = A(b\bar{\varepsilon})^{\theta}$$

   Again, note the correspondence to (8)
3. Inequality depends solely on the inequality the young's talents ε.
4. The assignment in (26) is a special case of the assignment in (24). The restriction in (26) follows from the linear homogeneity of ϕ that Proposition 2 invokes.

4.0.3 (w, π, V) on the BGP

We now solve for the prices that make the BGP allocations incentive compatible.

Proposition 3 *On the BGP wages and profits are*

$$\begin{aligned} w(x) &= \omega f(1, b\bar{\varepsilon})\, x & (32) \\ \pi(x) &= (1-\omega) f(1, b\bar{\varepsilon})\, x & (33) \end{aligned}$$

and lifetime utility is

$$V(y) = \left(\omega f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) + \beta(1-\omega)\frac{\Gamma}{b\bar{\varepsilon}}\right) y, \quad (34)$$

where the constant share of output paid to the young is

$$\omega = \frac{f_y(1, b\bar{\varepsilon}) + \beta\phi_y(1, b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma}{\frac{1}{b\bar{\varepsilon}} f(1, b\bar{\varepsilon}) + \beta\phi_y(1, b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma} \quad (35)$$

Proof. We proceed by construction. That is, we substitute the claimed prices into the Bellman equation (19), and show that they solve this equation as well as its FOC and that the solution for y of the FOC is then given by (26). This is the unique maximizing choice of y for the firm because its objective will be shown to be strictly concave in y. With the substitution from (32)-(34), (19) reads

$$(1-\omega) f(1, b\bar{\varepsilon}) x = \max_y \{f(x,y) - V(y) + \beta(1-\omega)(\phi(x,y))\} \tag{36}$$

with the FOC evaluated at the assignment in (26)

$$\begin{aligned} 0 &= f_y\left(1, \frac{y}{x}\right) - V_y + \beta(1-\omega)\phi_y\left(1, \frac{y}{x}\right) \\ &= f_y(1, b\bar{\varepsilon}) - \omega f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) - \beta(1-\omega)\phi\left(\frac{1}{b\bar{\varepsilon}}, 1\right) + \beta(1-\omega)\phi_y(1, b\bar{\varepsilon}) \end{aligned} \tag{37}$$

For any $\omega \in [0,1]$, the SOC holds in that the RHS of (36) is concave in y because V is linear in y, while and f and $(1-\omega)\phi$ are strictly concave in y. Concavity is true as long as $\omega \leq 1$. Evaluating the RHS of (19) at $y = b\bar{\varepsilon}x$, and subtracting $f(1, b\bar{\varepsilon})x$ from both sides, it reads

$$-\omega f(1, b\bar{\varepsilon}) x = -\omega f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) y - \beta(1-\omega)\frac{\Gamma}{b\bar{\varepsilon}} y + \beta(1-\omega)\Gamma x$$

noting that $f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) y = f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) b\bar{\varepsilon}x = f(1, b\bar{\varepsilon})x$ and that $\phi\left(\frac{1}{b\bar{\varepsilon}}, 1\right) y = \phi(1, b\bar{\varepsilon})x = \Gamma x$, and canceling x from both sides, (19) evidently holds for all x for any ω. Therefore we can just solve (37) for the one unknown, ω. Noting that $\phi\left(\frac{1}{b\bar{\varepsilon}}, 1\right) = \frac{1}{b\bar{\varepsilon}}\Gamma$, get

$$\omega = \frac{f_y(1, b\bar{\varepsilon}) + \beta\phi_y(1, b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma}{f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) + \beta\phi_y(1, b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma},$$

i.e., (35). ■

Cobb-Douglas example: Let $b\bar{\varepsilon} = 1$, and let f and ϕ both be Cobb-Douglas:

$$\left.\begin{array}{r} f(x,y) = x^{1-\rho}y^{\rho} \\ \phi(x,y) = Ax^{1-\theta}y^{\theta} \end{array}\right\} \Rightarrow \left\{\begin{array}{r} \Gamma = A \\ f_y(1,1) = \rho \\ \phi_y(1,1) = \theta A \end{array}\right. \tag{38}$$

Substituting into (35), we obtain

$$\omega = \frac{\rho - \beta(1-\theta)A}{1 - \beta(1-\theta)A}, \tag{39}$$

Remarks on Proposition 3

1. *Compensating differentials.*—As long as $\beta > 0$, wages include a compensating differential, and if this differential is large, wages can turn into tuition payments. This can happen if x matters a lot relative to y in both production and training. In the Cobb-Douglas example (38) leading to (39), $\omega < 0$ iff, $\rho < \beta A (1 - \theta)$, which reflects the intuition just mentioned. The compensating differential component of ω in (39) predictably goes to zero if $\beta \to 0$ in which case the young do not value the training that the firm provides, or if $\theta \to 1$, in which high-x firms are no better at providing training than low-x firms.

2. When $\omega < 0$, the young must pay tuition, and for this they would need an endowment, or a second asset like a bond on which the young could go short, or banks that would extend credit.[6] The equilibrium rate would be $\beta^{-1} - 1$. But both instruments (credit, short sales) require collateral in the real world, and without an endowment, the only collateral the young could offer would be having employment at the firm, and human capital is not a legally valid form of collateral. This problem arises here only when $\omega < 0$, but in Sec. 5.1 which treats the case in which y is private information, it always arises for a subset of the agents.

3. Note that even $\omega < 0$, the objective in (36) remains concave.

4. *The experience premium.*—Over an agent's lifetime wages grow faster than output. Wages in youth are a fraction ω of the output of the firm in which the young agent works. When the worker gets old, his firm's output has grown by the factor A, and he then receives a fraction $1 - \omega$ of that larger output. Therefore his wage growth and experience premium are

$$\text{total wage growth} = \frac{1-\omega}{\omega} A = \frac{1-\rho}{\rho - (1-\theta)\beta A} A, \tag{40}$$

$$\text{experience premium} = \frac{1-\omega}{\omega} A - A = \frac{1-2\omega}{\omega} A. \tag{41}$$

5 The BGP under private information

So far we assumed that x and y were public information. But either may be private information.[7] It turns out that a separating equilibrium still exists and that it supports the same assignment and the same BGP, the wages support it differ from (32).

[6] I thank A. Islamaj for pointing this out.

[7] I thank R. Lagunoff for pointing this out.

5.1 Private information about y

A separating equilibrium with positive sorting still survives if y is known privately, but now it requires that high-x firms pay lower wages regardless of the parameters. Wages are still paid up front and not contingent on the output produced. As long as $\theta > 0$, the worker's quality affects how much training is received, and this also depends on x, and this is an effective screening device because the high-y types would pay more to work with a high x type.[8]

Assume that x (and hence whatever x' that worker y ends up with) is public information. In this case wages must be decreasing in x regardless of the parameters, otherwise every worker would seek the highest-x job since that job would offer highest wages in youth and in old age. Write the wage as

$$w(x) = w_0 \bar{x} - wx. \tag{42}$$

Consider the decision problem of a worker of type y. He seeks to maximize his lifetime value over the choice of the type of firm x. Then, assuming that he will face the same linear assignment (31) in old age,

$$\hat{V}(y) = \max_x \{w_0 \bar{x} - wx + \beta(-w_0 \bar{x}\Gamma + [w + f(1, b\bar{\varepsilon})]\phi(x, y))\} \tag{43}$$

where (29) still holds so that $\Gamma = \phi(1, b\bar{\varepsilon})$. The problem is strictly concave in x. The FOC is

$$\begin{aligned} w &= \beta(w + f(1, b\bar{\varepsilon}))\phi_x(x, y) \\ &= \beta(w + f(1, b\bar{\varepsilon}))\phi_x(1, b\bar{\varepsilon}) \end{aligned}$$

and therefore at the uniquely maximizing assignment,

$$w = \frac{\beta f(1, b\bar{\varepsilon})\phi_x(1, b\bar{\varepsilon})}{1 - \beta\phi_x(1, b\bar{\varepsilon})}. \tag{44}$$

Proposition 4 *When y is private information, (α, H, Γ) still satisfy (26)-(29), but $w(x)$ is given in (42) in which the slope coefficient is given in (44) and in which w_0 is arbitrary*

Remarks:

1. The constant w_0 affects only the distribution of income between the old and the young, as well as the lifetime value of the young in (43), so let us just set $w_0 = 0$.

[8]The intuition goes back to at least Salop and Salop (1976) where the unobservable type was the propensity to quit a job, and where a stable worker prefer a backloaded contract.

2. $w'(x) = -w$ is the compensating differential per unit of x.

Cobb-Douglas example, again: Just as in (38), let $b\bar{\varepsilon} = 1$, and let f and ϕ both be Cobb-Douglas: $f(x,y) = x^{1-\rho}y^{\rho}$, $\phi(x,y) = Ax^{1-\theta}y^{\theta}$, so that $\Gamma = A$, $f(1,1) = 1$ and $\phi_x(1,1) = (1-\theta)A$. Substitute these values into (44) to get the compensating differential per unit of x:

$$w = \frac{\beta(1-\theta)A}{1-\beta(1-\theta)A} \tag{45}$$

Note that in this case, w is identical to the compensating differential component of ω in (39). The differential again goes to zero if $\beta \to 0$ because then the young do not value training, or if $\theta \to 1$, in which case high-x firms provide the same training as low-x firms.

5.2 Private information about x

We now show that if x is private information, a BGP with a separating equilibrium still exists, but the wage must now be higher for higher y types. The hypothetical BGP wage paid to a young type y is $w^{\mathrm{H}_t}(y) \equiv \Gamma^t w(y\Gamma^{-t})$. Since worker y matches with $b\bar{\varepsilon}x$, and since x grows by the factor Γ, under balanced growth, the worker's y would become $(b\bar{\varepsilon})^{-1}y\Gamma$ in the next period, and next period the output of his firm would equal

$$f\left[(b\bar{\varepsilon})^{-1}y\Gamma, y\Gamma\right] = f\left[(b\bar{\varepsilon})^{-1}, 1\right]y\Gamma$$

and the wage that he would pay to the young worker that he would then hire would be

$$\Gamma w\left(\frac{y\Gamma}{\Gamma}\right) = \Gamma w(y)$$

Therefore, in such an equilibrium, his lifetime value would be

$$V(y) = w(y) + \beta\Gamma\left[f\left[(b\bar{\varepsilon})^{-1}, 1\right]y - w(y)\right]$$

This lifetime value, however, will not play the same role in the decision of the firm that the value in (16) played when x was known

Consider now the problem of the old, i.e., the private information counterpart to the problem posed in (17)-(19). We simplify to the BGP version. The key difference, however, is that a worker of type y assumes that the firm hiring him is of type $(b\bar{\varepsilon})^{-1}y$ because that's what the equilibrium assignment is. Thus a worker that is paid $w(y)$ assumes he is being hired by firm $(b\bar{\varepsilon})^{-1}y$, and that as a result he will get earn his equilibrium lifetime value $V(y)$. The problem becomes one without a constraint of the type expressed by (16). Rather, firm x solves the unconstrained, strictly concave problem:

$$\max_y \{f(x,y) - w(y)\} \tag{46}$$

with the FOC (evaluated at the equilibrium assignment) equal to

$$w'(y) = f_y\left((b\bar{\varepsilon})^{-1} y, y\right) = f_y\left((b\bar{\varepsilon})^{-1}, 1\right).$$

which means that the equilibrium wage is

$$w(y) = C + f_y\left((b\bar{\varepsilon})^{-1}, 1\right) y \tag{47}$$

where C is a constant. We summarize this case as follows:

Proposition 5 *When x is private information, (α, H, Γ) still satisfy (26)-(29), but $w(x)$ is given in (47) in which C is arbitrary*

Remarks:

1. The constant C affects only the distribution of income between the old and the young.

2. The expression in (47) is increasing in y and since sorting is positive, high-x firms pay higher wages. Compared to the other two BGP wages in (32) and (42), this case displays the steepest relation between x and w, because $w(y)$ in (47) contains no compensating differential for training. Separation requires that this should be so.

Cobb-Douglas example once again: Again, let $b\bar{\varepsilon} = 1$, and let $f(x, y) = x^{1-\rho} y^{\rho}$, so that $f_y(1, 1) = \rho$. Then

$$w(y) = C + \rho y \tag{48}$$

6 Assignment frictions

Assignment frictions are assumed to originate in imperfect information about a worker's ability. Let $s =$ publicly observed signal of y.

$$\Pr(\tilde{s} \leq s \mid y) = F(s \mid y).$$

Then a young worker's type is s and wages and assignment depend on s. We match x to s using the distributions Φ and H as shown below:

$$\begin{aligned}
\Phi(s) &= \int F(s \mid y)\, dG(y) = \text{ signal distribution} \\
s &= \Phi^{-1}(H(x)) = \text{ assignment} \\
\pi(y \mid s) &= \text{ posterior} \\
\pi\left(y \mid \alpha^{-1}(x)\right) &= \text{ distribution of } y \text{ given } x \\
\Pr(\tilde{x} \leq x' \mid x) &= \Pr\left(A x^{1-\theta} y^{\theta} \leq x' \mid x\right) = \pi\left(\left(\frac{x'}{A x^{1-\theta}}\right)^{1/\theta} \mid \alpha(x)\right)
\end{aligned}$$

because $x' = Ax^{1-\theta}y^{\theta}$. Evolution of H:

$$H_{t+1}(x') = \int \pi \left(\left(\frac{x'}{Ax^{1-\theta}} \right)^{1/\theta} \mid \Phi_t^{-1}(H_t(x)) \right) dH_t(x)$$

Example: Log normal

$$\hat{s} = \hat{y} + \eta$$

where $E(\eta) = 0$. Recall that $y = b\bar{x}\varepsilon$. Let the squared correlation coefficient be

$$r^2 = \frac{\sigma_{\hat{y}}^2}{\sigma_{\hat{y}}^2 + \sigma_{\eta}^2} \equiv \text{ assignment quality}$$

Then

$$\hat{y}(\hat{s}) \sim N\left(r^2\hat{s} + \left(1 - r^2\right)\mu_{\hat{y}}, \quad \left(1 - r^2\right)\sigma_{\hat{y}}^2\right) \tag{49}$$

Then the evolution of x is

$$\hat{x}' = \hat{A} + (1-\theta)\hat{x} + \theta\hat{y}(\alpha[x]) \tag{50}$$

where

$$\hat{s} = \hat{\alpha}(x) = \mu_{\hat{s}} + \frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}(\hat{x} - \mu_{\hat{x}}) \tag{51}$$

The distribution of $\hat{y}$ conditional on $\hat{x}$:

$$\hat{y}(\hat{\alpha}(x)) \sim N\left(\mu_{\hat{y}} + r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}(\hat{x} - \mu_{\hat{x}}), \quad \left(1 - r^2\right)\sigma_{\hat{y}}^2\right) \tag{52}$$

Substituting into (50),

$$\hat{x}' = \hat{A} + \theta\left(\mu_{\hat{y}} - r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}\mu_{\hat{x}}\right) + (1-\theta)\hat{x} + \theta r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}\hat{x} + \sqrt{(1-r^2)}\sigma_{\hat{y}}\varepsilon \tag{53}$$

where ε ˜ $N(0,1)$, so that

$$\hat{x}' - \hat{x} = \hat{A} + \theta\left(\mu_{\hat{y}} - r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}\mu_{\hat{x}}\right) + \theta\left(r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}} - 1\right)\hat{x} \tag{54}$$

Now

$$r^2\sigma_{\hat{s}} = \frac{\sigma_{\hat{y}}^2}{\sigma_{\hat{y}}^2 + \sigma_{\eta}^2}\sqrt{\sigma_{\hat{y}}^2 + \sigma_{\eta}^2} = r\sigma_{\hat{y}},$$

and therefore

$$\hat{x}' - \hat{x} = \hat{A} + \theta\left(\mu_{\hat{y}} - r\frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}}\mu_{\hat{x}}\right) + \theta\left(r\frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}} - 1\right)\hat{x} \tag{55}$$

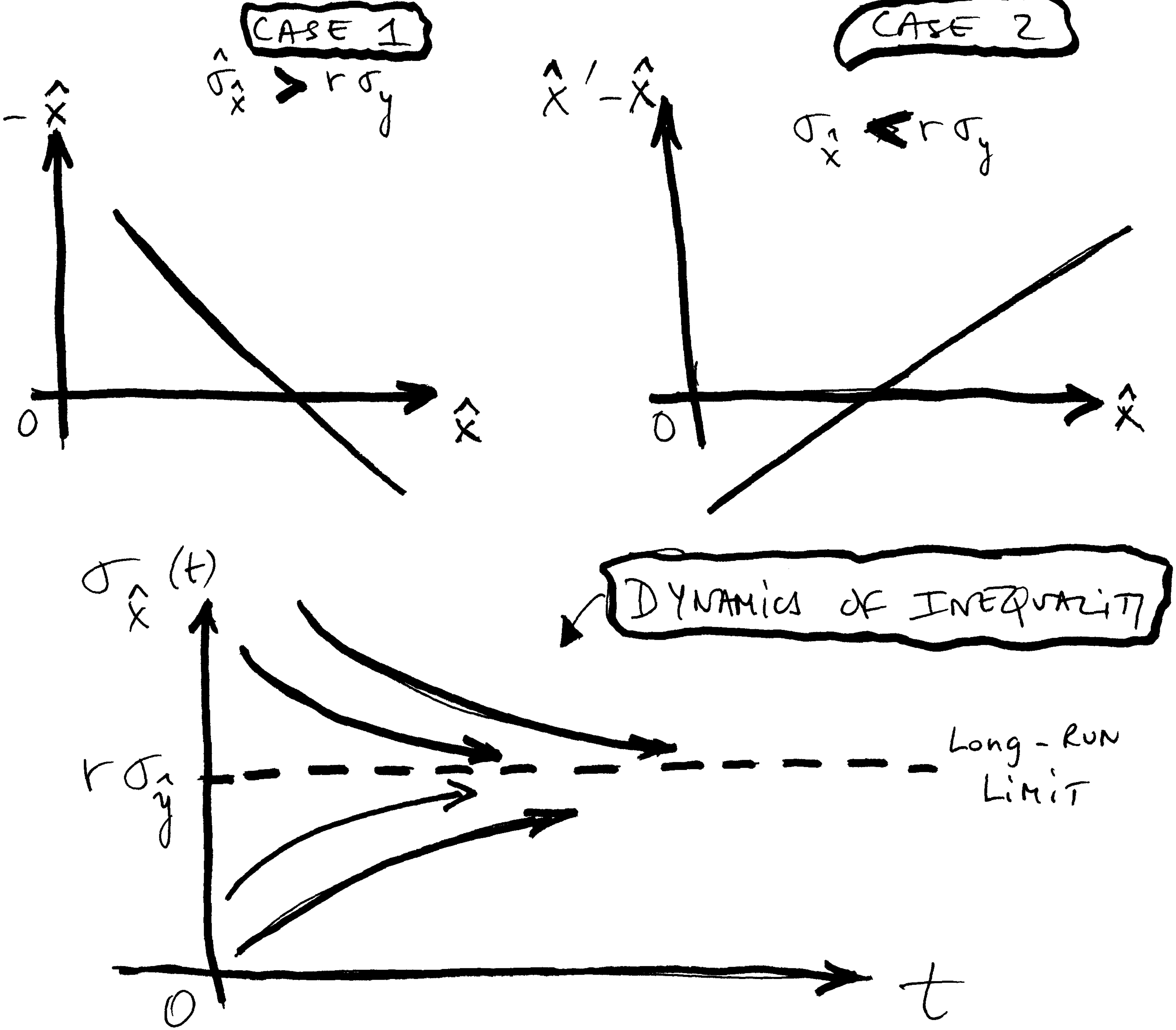
CASE 1
$\hat{\sigma}_{\hat{x}} > r\sigma_y$
$\hat{x}' - \hat{x}$
0
$\hat{x}$
CASE 2
$\sigma_{\hat{x}} < r\sigma_y$
$\hat{x}' - \hat{x}$
0
$\hat{x}$
DYNAMICS OF INEQUALITY
$\sigma_{\hat{x}}(t)$
$r\sigma_{\hat{y}}$
LONG-RUN LIMIT
0
t

FIGURE 1

6.1 Dynamics of inequality and the convergence to Gibrat's Law

This discussion is accompanied by Figure 1. From (55) we find that

$$\frac{\partial}{\partial x}\left(\hat{x}'-\hat{x}\right) \gtreqless 0 \qquad \text{as} \qquad \sigma_{\hat{x}} \lesseqgtr r\sigma_{\hat{y}}. \tag{56}$$

Case 1 of Figure 1 depicts what happens if inequality in $\hat{x}$ exceeds $r\sigma_{\hat{y}}$. In that case, equation (56) states that the low-x firms experience faster growth than the high-$\hat{x}$ firms, and this causes $\sigma_{\hat{x}}$ to decline. The opposite happens in Case 2 in which inequality in $\hat{x}$ is below $r\sigma_{\hat{y}}$. In that case it is the high-x firms that grow faster, thereby raising $\sigma_{\hat{x}}$. Moreover,

$$\frac{\partial}{\partial x}\left(\hat{x}'-\hat{x}\right) > (<)\, 0 \Rightarrow \sigma_{\hat{x}} \text{ is increasing (decreasing) with } t,$$

This is illustrated in the bottom panel of Figure 1. It shows that in the long run we converge to the limit in which $\frac{\partial}{\partial x}\left(\hat{x}'-\hat{x}\right)=0$ so that Gibrat's Law holds, and at that point

$$\sigma_{\hat{x}} = r\sigma_{\hat{y}}. \tag{57}$$

Time is measured in units of one half-working-life, i.e., convergence is slow. In the OG model, Kremer and Thomson (1994) already noted that convergence will be slow.

6.2 Garbling and inequality

The model implies a positive long-run relationship between development and inequality. When r declines, inequality shrinks, because positive assignment provides leverage to a high x.[9] In the models of Restucchia and Rogerson (2008) and Hsieh and Klenow (2009), the same result obtains in that that an efficient firm will, when scale is variable, choose to operate at a higher scale and therefore raise its profits by more than if it would had its scale remained unchanged, and this force is attenuated when there is misallocation. As a theory of income inequality, this theory provides a rationale for the upward-sloping portion of the Kuznets curve, but not a downward portion. The evidence in favor of a rising inequality with development curve is mixed, with the time series showing a mildly positive relation (Frazer 2006, Table 2) and the cross-section mildly supporting the inverted-U relation.

As $r \Rightarrow 0$, $\sigma_{\hat{x}} \to 0$, but the absolute standard deviation also goes to infinity because x does, but more slowly than x. Since a fall in r also has a negative long-run effect on development, the positive relation between development and inequality emerges.

[9] A similar inequality result is also in models of marital sorting (Kremer 1997, Fernandez and Rogerson 2001), and apparently evidence supports it.

In the transition, Figure 1 showed that inequality can rise or fall, depending on where it is relative to its steady state value. The most reasonable time-series experiment is a sudden rise in r due, perhaps, to a policy reform that allows quality of the young to be better labelled and more appropriately rewarded. In that case, the horizontal line in the bottom panel of Figure 1 would rise, and the economy would find its inequality and its development rising along the transition to the new steady state. In this sense, the model can generate a rising inequality in the time-series sense.[10]

6.3 Long-run development gaps

Log output is

$$\hat{q} = (1-\rho)\,\hat{x} + \rho\hat{y}.$$

Distribution of output conditional on x.—From (52), log output of firm x is distributed as $N\left(\mu_{\hat{q}(x)}, \sigma^2_{\hat{q}(x)}\right)$ where

$$\begin{aligned}\mu_{\hat{q}(x)} &= (1-\rho)\,\hat{x} + \rho\left(\mu_{\hat{y}} + r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}\left(\hat{x} - \mu_{\hat{x}}\right)\right)\\ \sigma^2_{\hat{q}(x)} &= \left(1-r^2\right)\rho^2\sigma^2_{\hat{y}}\end{aligned}$$

output is unconditionally distributed as $N\left(\mu_{\hat{q}}, \sigma^2_{\hat{q}}\right)$, where

$$\begin{aligned}\mu_{\hat{q}} &= (1-\rho)\,\mu_{\hat{x}} + \rho\mu_{\hat{y}}\\ \sigma^2_{\hat{q}} &= \left(1-\rho+\rho r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}}\right)^2\sigma^2_{\hat{x}} + \left(1-r^2\right)\rho^2\sigma^2_{\hat{y}}\end{aligned}$$

Since

$$r^2\frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}} = r^2\frac{\sqrt{\sigma^2_{\hat{y}}+\sigma^2_{\eta}}}{\sigma_{\hat{x}}} = r^2\frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}}\sqrt{\frac{\sigma^2_{\hat{y}}+\sigma^2_{\eta}}{\sigma^2_{\hat{y}}}} = r\frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}}, \tag{58}$$

we have

$$\begin{aligned}\sigma^2_{\hat{q}} &= \left(1-\rho+\rho r\frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}}\right)^2\sigma^2_{\hat{x}} + \rho^2\left(1-r^2\right)\sigma^2_{\hat{y}}\\ &= (1-\rho)^2\sigma^2_{\hat{x}} + \rho^2 r^2\sigma^2_{\hat{y}} + 2\rho(1-\rho)\,r\sigma_{\hat{y}}\sigma_{\hat{x}} + \rho^2\left(1-r^2\right)\sigma^2_{\hat{y}}\\ &= (1-\rho)^2\sigma^2_{\hat{x}} + 2\rho(1-\rho)\,r\sigma_{\hat{y}}\sigma_{\hat{x}} + \rho^2\sigma^2_{\hat{y}}\end{aligned}$$

Thus interchanging $(1-\rho, \sigma_{\hat{x}})$ and $(\rho, \sigma_{\hat{y}})$ leaves the value of $\sigma^2_{\hat{q}}$ unchanged.

[10] Inequality may be bad for growth for other reasons, reasons not included in the model. Easterly (2004) finds some evidence for a causal negative effect of inequality on development.

Compare two economies starting with the same $x \sim N\left(\mu_{\hat{q}}, \sigma_{\hat{q}}^2\right)$ but with a different r, say r_1 and r_2. Then the ratio of the two economies' aggregate outputs is

$$\frac{Q_1}{Q_2} = \exp\left(\rho\left(1-\rho\right)\sigma_{\hat{y}}\sigma_{\hat{x}}\left[r_1 - r_2\right]\right) \tag{59}$$

6.3.1 Allocative efficiency and development gaps

1. Inequality $\uparrow$ as high-x more easily match with high-y. As (57) shows, garbling reduces inequality and, indeed, as $r \to 0$, so does inequality of log incomes in the sense that the coefficient of variation of the distribution of H_t converges to zero as $\log x$ converges to infinity.

2. Output $\uparrow$ and growth $\uparrow$:

$$\frac{\partial}{\partial r}\left(\hat{x}' - \hat{x}\right) \gtreqless 0 \qquad \text{as} \qquad \hat{x} \gtreqless \mu_{\hat{x}}$$

Aggregate output is

$$Q^{\mathrm{H}} \equiv \int f\left(x, y\right) d\tau\left(y \mid \alpha\left(x\right)\right) dH\left(x\right) = \int f\left(x, y\right) d\tau\left(y \mid \alpha\left(x\right)\right) dH\left(x\right)$$

Consider the long-run gap between two economies with different values for r. In the long run, (57) holds, and it states that $\sigma_{\hat{x}} = r\sigma_{\hat{y}}$. Substituting into (59), the steady-state output ratio is

$$\lim_{t\to\infty} \frac{Q_{1,t}}{Q_{2,t}} = \exp\left(\rho\left(1-\rho\right)\sigma_{\hat{y}}^2\left[r_1^2 - r_2^2\right]\right). \tag{60}$$

Figure 2 plots this as a function of r^2 for various values of the parameter combination $\rho\left(1-\rho\right)\sigma_{\hat{y}}^2$, assuming that one of the economies has perfect assignment:

Development gap relative to leader.—Suppose the leader has $r = 1$. Then the steady state development gap is

$$1 - \exp\left(-\left(1 - r^2\right)\rho\left(1-\rho\right)\sigma_{\hat{y}}^2\right). \tag{61}$$

Since $\rho\left(1-\rho\right) \leq 1/4$, the largest gap that the model can explain is $1 - \exp\left(-\frac{1}{4}\sigma_{\hat{y}}^2\right)$

$$0.05 \leq 1 - \exp\left(-\frac{1}{4}\sigma_{\hat{y}}^2\right) \approx 0.08 \leq 0.11.$$

which is based on the inequalities $0.2 \leq \sigma_{\hat{y}}^2 \approx 0.33 \leq 0.47$ taken from Heathcote *et al.* (2005) where in Fig. 1A they report the variance of log earnings at various ages. The heavy line in Fig. 1 assumes that 1/2 and that $\sigma_{\hat{y}}^2 = 0.33$, the latter being the most

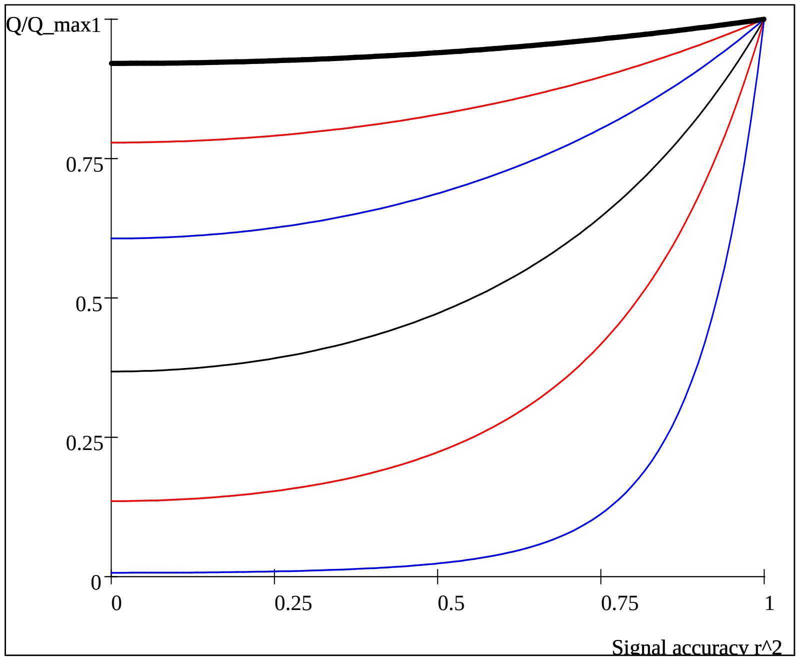

Figure 2: Development gap and r^2 for $\rho(1-\rho)\,\sigma^2_{\hat{y}} \in \left\{\frac{.33}{4}, \frac{1}{4}, \frac{1}{2}, 1, 2, 5\right\}$

reasonable estimate of this parameter based on the data in the paper by Heathcote *et al.*.

The exercise in Figure 2 assumes that the economies being compared are closed and that H refers to the distribution of x in a given economy. An extension would regard the spillover in (21) as including an inflow of knowledge from abroad which would seem reasonable. Such spillovers would probably reduce the model's ability to explain world inequality. At some level the model would be suited for explaining within-country as well as cross-country inequality as in documented in Sala-i-Martin (2006).

6.3.2 Long-run distribution of output and wages

Steady-state wages and profits again are

$$w(x) = \omega x, \qquad \text{where} \qquad \omega = \frac{\rho - \beta A(1-\theta)}{1 - \beta A(1-\theta)}$$

so that the distribution of incomes is proportional to the distribution of x.

6.4 Wages and profits in the transition

THIS SECTION IS UNDER CONSTRUCTION

Distribution of output among firms depends on how teams combine into firms. If

NB: Initial H_0 must be log-normal – thus, only a subset of the full dynamics are analyzed

How does the functional distribution of incomes, i.e., the share of labor, shift over the transition path?

Recall: $q = x^{1-\rho} y^{\rho}$ and $x' = Ax^{1-\theta} y^{\theta}$. Since

Assignment is log linear: $s = Cx^{\sigma_{\hat{s}}/\sigma_{\hat{x}}}$, i.e.,

$$\hat{s} = \mu_{\hat{s}} + \frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}} (\hat{x} - \mu_{\hat{x}})$$

so that

$$\begin{aligned} \hat{y} &= \left(1 - r^2\right) \mu_{\hat{y}} + r^2 \hat{s} + \sqrt{(1 - r^2)} \sigma_{\hat{y}} \zeta \\ &= \text{const. } + r^2 \frac{\sigma_{\hat{s}}}{\sigma_{\hat{x}}} x + \text{noise} \end{aligned}$$

where $\zeta \,\tilde{}\, N(0, 1) \Rightarrow$

out of steady state $y = \alpha(x)$ is convex or concave in x

output is convex or concave in x

Wages that firm x pays, $w(x)$, are up front

Equilibrium lifetime utility is:

$$V(s) = w\left(\alpha^{-1}(s)\right) + \beta \int \pi\left(\phi\left[\alpha^{-1}(s), y\right]\right) d\tau(y \mid s) \tag{62}$$

Firm x's decision problem.

$$\pi(x) = \max_{w,s} \left\{ \int f(x, y) \, d\tau(y \mid s) - w \right\} \tag{63}$$

s.t.

$$w + \beta \int \pi(\phi[x, y]) \, d\tau(y \mid s) \geq V(s) \tag{64}$$

leading to the consolidated problem

$$\pi(x) = \max_{s} \left\{ -V(s) + \int [f(x, y) + \beta \pi(\phi[x, y])] \, d\tau(y \mid s) \right\} \tag{65}$$

Try a profit function of the isoelastic form

$$\pi(x) = \pi_0 x^{\pi} \tag{66}$$

$$\pi_0 x^{\pi} = \max_{s} \left\{ -V(s) + \int [f(x, y) + \beta \pi_0 \phi(x, y)^{\pi}] \, d\tau(y \mid s) \right\} \tag{67}$$

to be continued.

7 Transition dynamics with no frictions

The previous section solved completely for the full dynamics when the distributions and the noise are log normal. The initial condition had to be log normal, however, and this then is only a small subset of all the initial conditions that one could consider. We now set up the notation for the dynamics when there are no frictions; this notation can accommodate any initial conditions, but it will not allow us to solve explicitly

Although with no frictions the BGP is unique, we have not yet shown that the BGP is stable even locally. The state of the system, H, is an infinite-dimensional object, and because it keeps shifting to the right, H needs to be normalized before we can discuss the properties of a fixed point; (29) states the growth factor $\Gamma = \phi(1, b\bar{\varepsilon})$ that we can use to normalize the map χ in (14) and seek to show that it is a contraction in the normed space of continuous C.D.F.s on the line.

Let us shift temporarily to the time domain. If $z_t = \Gamma^{-t} x_t$ and if, as stated by (13), $x_{t+1} = \xi_t x_t$ then i.e., $z_{t+1} = (\Gamma^{-1}\xi_t) z_t$. Our normalized aggregate law of motion, call it $\tilde{\chi}$, should map the set of C.D.F.s on the positive line into itself. Denote by $F(z)$ the C.D.F. of z. We then seek to show that the fixed point to the map

$$F' = \tilde{\chi}(F)$$

(meaning$F'(z') = F\left[\Gamma(\xi)^{-1}(z')\right]$).is unique and globally stable. Stated compactly, the aggregate law of motion is

$$\tilde{\chi}(F) = F \cdot \left(\Gamma^{-1}\xi\right)^{-1}.$$

TO BE CONTINUED

The next section describes, partially, the dynamics when there are frictions, and the frictionless case is almost certainly the limit as the friction disappears.

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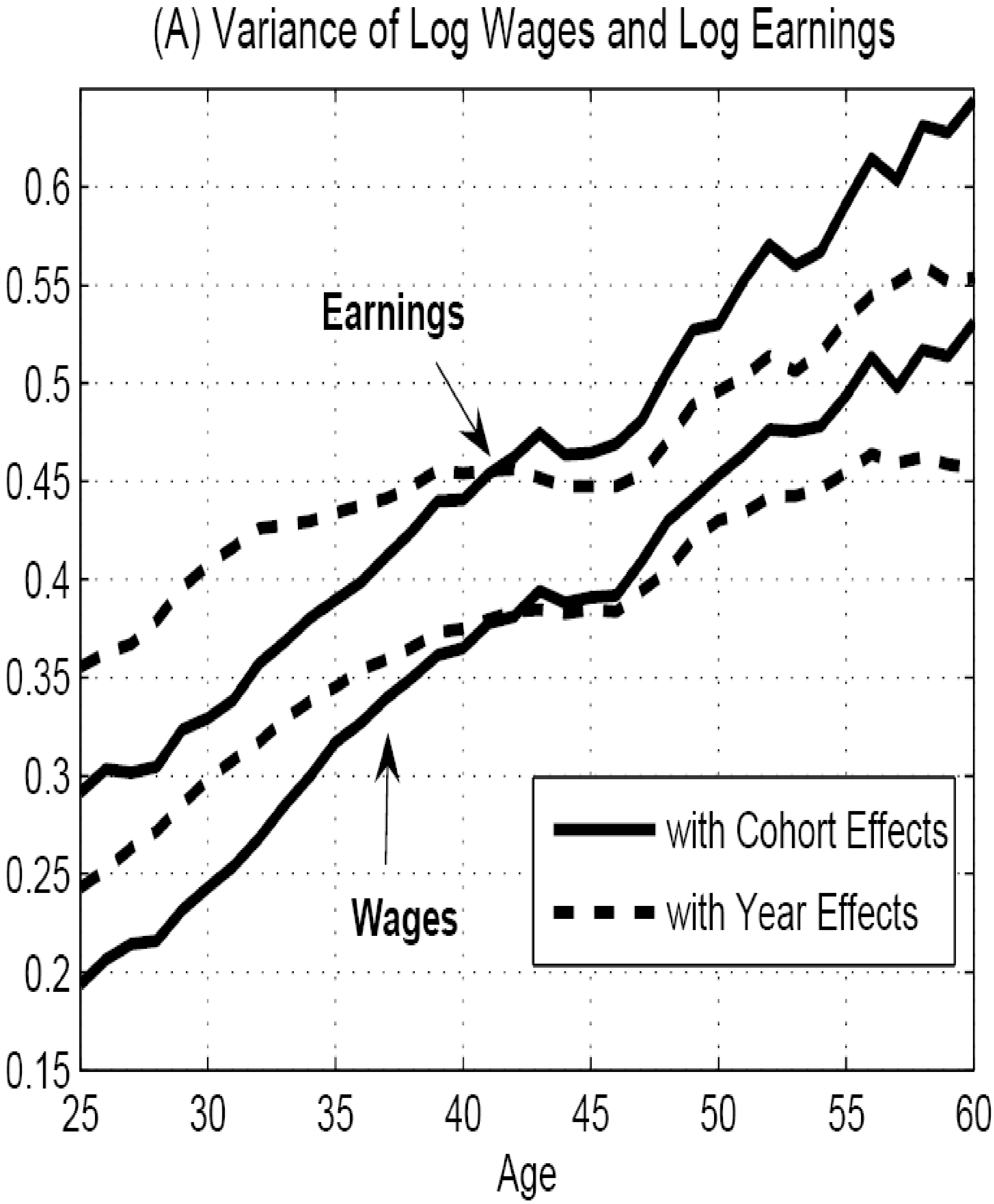

Figure 3: Figure 1A from Heathcote, Storesletten and Violante (2004)