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Crises: Equilibrium Shifts and Large Shocks

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Crises: Equilibrium Shifts and Large Shocks*

Stephen Morris and Muhamet Yildiz

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Abstract

A coordination game with incomplete information is played through time. In each period, payoffs depend on a fundamental state and an additional idiosyncratic shock. Fundamentals evolve according to a random walk where the changes in fundamentals (namely common shocks) have a fat tailed distribution. We show that majority play shifts *either* if fundamentals reach a critical threshold *or* if there are large common shocks, even before the threshold is reached. The fat tails assumption matters because it implies that large shocks make players more unsure about whether their payoffs are higher than others. This feature is necessary for large shocks to matter.

1 Introduction

On July 26th, 2012, Mario Draghi gave a speech in which he promised "...to do whatever it takes to preserve the euro. And believe me, it will be enough...". The "whatever it takes..." speech has been widely credited for shifting the Eurozone economy from a "bad equilibrium," with high sovereign debt spreads and growing fiscal deficits mutually reinforcing each other; to a "good equilibrium"—with low spreads and sustainable fiscal policy. But why did the speech switch the equilibrium? Brunnermeier, James, and Landau (2016) note that "...the response—immediately convincing markets that the tide had been turned—may look curious." There had been many speeches with many policy announcements, all

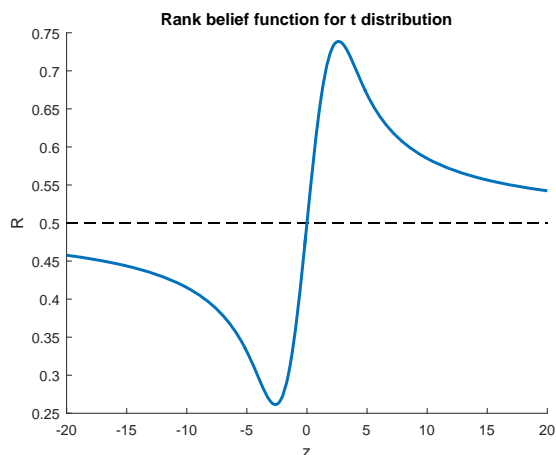
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carefully designed to turn the tide, that had not succeeded. It was surely the case that the bad equilibrium continued to exist: if market participants had continued to short sovereign debt, spreads would have remained high, undermining fiscal stability, and so on in a self-fulfilling way. Our purpose in this paper is to develop a theoretical framework to develop insights into when equilibrium shifts occur—from "bad" to "good" as in this case—or from "good" to "bad".

Our analysis takes place in a canonical model of coordination, but with dispersed information through time. In each period, each player must decide whether to invest or not. His return to investing is increasing in both his own *payoff type* and the proportion of others investing. If his type is high enough, he has a dominant strategy to invest. If his type is low enough, he has a dominant strategy to not invest. For intermediate values, he will invest only if he expects others to invest. Each player's type is equal to a fundamental state ("fundamentals") plus an idiosyncratic shock. Fundamentals evolve according to a random walk; we will refer to the innovation in the fundamentals as a common shock. In each period, there is common knowledge of the previous level of the fundamentals; each player knows his own type, but cannot decompose the difference between this payoff type and the previous level of fundamentals into common and idiosyncratic components. Common shocks are drawn from a fat-tailed distribution, i.e., one with a power law distribution of the tail density; idiosyncratic shocks are drawn from a distribution with thinner tails. See discussion below for further discussion of these assumptions.

Our main results will concern (super-)majority behavior in any (public perfect Bayesian Nash) equilibrium. There will be a *fundamental investment threshold*, such that majority behavior will shift to investment whenever fundamentals move above that threshold. (The model will be symmetric so analogous results hold throughout for no investment). We will provide a characterization of the fundamental investment threshold and show that majority investment will be triggered before investment becomes a dominant strategy. But even if the fundamental state is below the fundamental investment threshold, a positive shock to fundamentals will trigger a switch to majority investment if fundamentals are above a *latency threshold* and the shock is sufficiently large. We will characterize the latency threshold and determine how large the shock must be to trigger a shift before the fundamental investment threshold is reached.

Our results reflect the importance of *rank beliefs* in coordination environments. How

Figure 1: Rank belief function R .

likely does a player i think it is that a representative other player will have a lower type than him? Equivalently, what is his expectation of the proportion of other players who will have lower types? We call this the rank belief of player i ; it is the expectation of the percentile of player i 's type conditional on his type. We plot a typical rank belief function in Figure 1, where the difference between a player's type and the prior fundamentals is plotted on the horizontal axis. This particular rank function arises when common shocks have a Student t -distribution and the idiosyncratic shocks have a normal distribution—an example we will be using throughout the paper—but the shape illustrates general properties under our maintained assumptions of fat common shocks and thinner idiosyncratic shocks. Assuming that common and idiosyncratic shocks have symmetric distributions, the rank belief is $1/2$ when the type is equal to prior fundamentals. The rank belief then increases as the player is higher type, because of reversion to the mean: he attributes some of his higher type to his idiosyncratic shock and he has no information about the other player's idiosyncratic shock. But as his type gets very large, he attributes it almost entirely to the common shock, because of the fat-tail assumption, resulting in rank belief near $1/2$. There is a compelling intuition for such non-monotone behavior. Recall that the t -distribution arises when the common shocks are normally distributed with unknown variance (drawn from inverse χ^2 distribution). In deciding how much to attribute to common shock, he also updates his belief about the variance of the common shock. As he observes larger deviations, he comes to believe that the variance is larger and attributes a larger share to the common

shock.

These properties of rank beliefs generate our results. Since invest is dominant for large payoff types, there is a largest type for whom not invest is rationalizable; call that type *marginal*. The marginal type can assign at most his rank belief to other players' not investing. Armed with this observation, we can explain our main results using Figure 1. There is an upper bound \bar{R} to rank beliefs (approximately 0.735 in Figure 1), and there is a threshold on payoffs above which invest is the only best response if the fraction of players who are not investing is at most \bar{R} . Invest is uniquely rationalizable above that threshold because the rank belief of the marginal type can be at most \bar{R} . This explains the fundamental investment threshold and our "level" result.

Now consider our "large shock" result. Fix any rank belief r , where $1/2 < r < \bar{R}$. We see from Figure 1 that there will be a critical type with the property that all higher types will have a rank belief of at most r . But now as long as invest is the only best response for that critical type when he assigns probability at most r to other players not investing, invest will be uniquely rationalizable for that type: if the marginal type were greater than the critical type, we would have a contradiction because that type would also assign probability at most r to other players not investing. But now there is a large enough shock that induces that player to invest even though the fundamental investment threshold has not been reached. This large shock can work only if $r > 1/2$. Defining the *latent* action to be the optimal action for a player whose expectation of the proportion of others investing is $1/2$, the large enough shock will matter only if makes invest the latent action. Now the latency threshold described above is the payoff type where the latent action switches from not invest to invest.

Our analysis offers a distinctive interpretation of the impact of Draghi's speech. We interpret Draghi's speech as a "large shock". But it was not so large as to move the fundamentals substantially, let alone pushing them over the fundamental investment threshold. Rather, it was large enough for types of market traders who might be inclined to hold negative positions in sovereign debt to assign higher probability to others being higher types who were less inclined to short. As such, it shifts the equilibrium by creating confusion and breaking common certainty about the fundamentals. As outlined above, this argument relies on the good equilibrium already being the latent equilibrium.

We identify conditions under which equilibrium must shift under *all* equilibria. When there are multiple equilibrium actions, an equilibrium can condition on history arbitrarily by

selecting any of the equilibria of the stage game, resulting in equilibrium shifts for no obvious reason. In order to avoid such spurious equilibrium shifts, we also propose an equilibrium with hysteresis, in which the majority behavior does not change unless it becomes unsustainable as an equilibrium behavior, similar to the hysteresis equilibrium under complete information (see Cooper (1994)). Our analyses provides a characterization of equilibrium shifts under the hysteresis equilibrium, where shifts occur only in the scenarios discussed above.

Our result about large shocks goes through only if the rank belief is non-monotone and, in particular, approaches $1/2$ as the type increases. These properties follow from our fat common tails / thinner idiosyncratic tails assumptions. These properties do not hold in general. For example, if both common shocks and idiosyncratic shocks are normally distributed, rank beliefs are monotone and converge to 1 as types increase. In this case, there is a fundamental investment threshold, but it will not be bounded away from the dominant strategy trigger and there will be no distinctive role for large shocks to trigger switches in majority behavior. It is important that larger positive shocks generate lower probabilities that others have smaller shocks, i.e., rank beliefs have the shape of Figure 1.

Our motivation for such rank beliefs is our desire to understand the role of model uncertainty. Here, the model can be thought of the distribution of the common shock, which affects everybody. As in the example of the t -distribution discussed above, uncertainty about the variance of the distribution of the common shock easily leads to fat tails: observing a large unexpected shock under the current model, players drastically update their (belief about the) model.

Our fat-tail assumption is also in line with a long-standing empirical literature that establish that the changes in key economic variables have fat tailed distribution, going back to Pareto's (1897) observation about income distribution (see surveys by Benhabib and Bisin (2016), Gabaix (2009) and Ibragimov and Prokharov (2016)). For example, the changes in prices, asset returns and foreign exchange rates all have fat tailed distribution (see pioneering works of Mandelbrot (1963) and Fama (1963), as well as contemporary studies such as Cont (2001) and Gabaix, Gopikrishnan, Plerou, and Stanley (2006)). Moreover, many commonly used theoretical models, such as GARCH models and models with stochastic volatility, naturally lead to fat tailed changes in the fundamental, as in the example of Student t -distribution above. In our model, individual payoff types are more volatile than the fundamentals as they

contain additional idiosyncratic shocks. Motivated with model uncertainty, we assumed that idiosyncratic shocks have thinner tails, so that the tails of the changes in payoff types are as thick as the tails of common shocks. This is similar to the fact that the tails of stock and market returns are empirically indistinguishable from each other. Idiosyncratic variation can also be interpreted noise in players' signals of the fundamental (see Remark 1) and, under this interpretation, thinner idiosyncratic tails corresponds to a well understood (if noisy) observation technology. It is our distributional assumptions, as well as our relaxation of the assumption that there is always a unique rationalizable play, that distinguishes our results from the existing first generation of global games models; we discuss this in length in Section 8 at the end of the paper.

We next introduce our model. In Section 3, we define and characterize the rank belief functions that will drive our results. In Section 4, we analyze the one-shot game, where we fix the fundamentals from the previous period and analyze behavior in the next period only. We characterize when one of the actions is "majority uniquely rationalizable"—that is, when we can show that invest, say, is the unique rationalizable action for a majority as a function of the current and prior fundamentals. This section delivers the key analysis of the paper. In Section 5, we formally analyze the dynamic model, describing strategies and the solution concept. We show when actions must be played in equilibrium independent of history and therefore when majority shifts must arise. In Section 6, we discuss majority shifts in the hysteresis equilibrium in greater detail. In Section 7, we review what happens if we relax the assumption that common shocks have a fat-tailed distribution. We discuss our broader contribution to the global games literature in Section 8. Some proofs are relegated to the Appendix.

2 Model

We have a continuum of players $i \in N = [0, 1]$ and discrete time with dates $t = 0, 1, 2, \dots$. For fixed real numbers θ_{-1} and $\sigma > 0$, at each t , the following happen in the given order:

1. a new state θ_t is drawn according to

$$\theta_t = \theta_{t-1} + \sigma \eta_t; \tag{1}$$

2. each player i observes his own payoff parameter x_{it} where

$$x_{it} = \theta_t + \sigma \varepsilon_{it}; \quad (2)$$

3. simultaneously, each player i chooses an action $a_{it} \in A \equiv \{\text{Invest, Not Invest}\}$ and enjoys stage-payoff of

$$u(\alpha_t, x_{it}) = x_{it} + \alpha_t - 1 \quad (3)$$

if he invests and 0 otherwise where α_t is the fraction of individuals who invest;

4. and finally, θ_t and α_t are publicly observed.

Here, the state θ_t is a random walk, and each player's payoff parameter x_{it} contains an additive noise term $\sigma \varepsilon_{it}$. Hence, the deviation $x_{it} - \theta_{t-1}$ consists of two components: a *common shock* $\sigma \eta_t$, which is common to both players, and an *idiosyncratic shock* $\sigma \varepsilon_{it}$. We will sometimes refer to θ_t as the fundamental state, or fundamentals. We write

$$z_{it} = \eta_t + \varepsilon_{it} = (x_{it} - \theta_{t-1}) / \sigma \quad (4)$$

for the normalized shock to the payoff of player i . We assume that ε_{it} and η_t are independently drawn—both serially and across the players—from distributions F and G , respectively, with positive densities f and g everywhere on real line; and that f and g are continuous and even functions that are weakly decreasing on \mathbb{R}_+ . We also assume that g has thicker tails:

Assumption 1 (fat tails) *The distribution g of common shocks has regularly-varying tails,*

$$\lim_{\lambda \rightarrow \infty} \frac{g(\lambda \eta)}{g(\lambda \eta')} \in (0, \infty) \text{ for all } \eta, \eta' \in \mathbb{R}_+, \quad (5)$$

and the distribution of idiosyncratic shocks has thinner tails

$$\lim_{\lambda \rightarrow \infty} \frac{f(\lambda \varepsilon)}{g(\lambda \eta)} = 0 \text{ for all } \varepsilon, \eta \in \mathbb{R}_+. \quad (6)$$

This is our main assumption. This has two parts. First, the distribution of common shocks has thicker tails than the usual distributions with exponential tails, such as the normal and exponential distributions. This is formalized by (5), which states that g has regularly-varying (i.e. fat) tails, as in Pareto and t -distributions. Second, the common shock distribution has thicker tails than idiosyncratic shock distribution (which may or may not have fat tails). This is formalized in (6).

We will write $F_{\theta|x}(\theta_t|x_{it},\theta_{t-1})$ for the conditional cumulative distribution function of θ_t given x_{it} and θ_{t-1} . Note that $F_{\theta|x}$ is translation invariant, depending only on the deviations from θ_{t-1} : $F_{\theta|x}(\theta_t + \Delta|x_{it} + \Delta, \theta_{t-1} + \Delta) = F_{\theta|x}(\theta_t|x_{it}, \theta_{t-1})$. For tractability, we make the following monotonicity assumption.

Assumption 2 (monotonicity) $F_{\theta|x}(\theta_t|x_{it},\theta_{t-1})$ is decreasing in x_{it} .

This assumption is made throughout the literature and is satisfied under usual specifications. It would have failed if Assumption 1 failed and the tails of idiosyncratic shocks were thicker than the tails of the common shock. Intuitively, Assumption 1 should make Assumption 2 easier to satisfy.

Since we have a continuum of players, in equilibrium, α_t and θ_t are sufficient statistics for each other. Hence, observability of either of them is enough. Here, we assume direct observation of θ_t for clarity. Following the repeated games literature, we assume that α_t is observable, but observability of α_t per se is not relevant for our analysis. The key assumption here is that no individual action is observed, and hence a single player's action does not have any impact on other players' future behavior. Therefore, the players choose myopic best replies, and their time preferences do not matter for our analysis.

Since θ_{t-1} is publicly observed at the end of period $t - 1$, all of the previous private information becomes obsolete and the players play a one-shot game. We will write $G(\theta_{-1})$ for the resulting one-shot game. When analyzing the one-shot game, we write θ_{-1} for the state from the previous period and drop other time indices. Because strategic analysis reduces to a one-shot game in any given period, it is convenient to first characterize individual and majority behavior in the one-shot game (in Section 4) and use that characterization to analyze equilibrium in the dynamic game (in Section 5). The analysis of the one-shot game provides the key analytic insights about strategic behavior of the paper. The analysis of the dynamic model will formalize our dynamic interpretation of our results, and allow us to identify distinctive implications of equilibrium selection based on hysteresis. But, in the next Section, we first report how our assumptions about the distribution of common and idiosyncratic shocks translate into rank belief properties that drive our main results about large shocks.

We now introduce some useful terminology. Each player i has a strictly dominant strategy to invest if x_i is strictly more than one, has a strictly dominant strategy to not invest if $x_i < 0$ and otherwise no action is strictly dominated. We will therefore refer $[0, 1]$ as *the*

undominated region and 0 and 1 as *the dominance triggers*. Under complete information (i.e., $\sigma = 0$), there are multiple equilibria in the region $[0, 1]$: all invest and all not invest. Game theoretic analysis suggest *refinements* to select among equilibria. We will say that invest is the *latent* action (at θ) when $\theta > 1/2$; and that not invest is the latent action when $\theta < 1/2$. The latency threshold is then $1/2$.¹

Remark 1 *Most global games analysis is carried out in the "common value" case, where players care about a common state of the world. The analysis in this paper corresponds to a "private value" global game (Morris and Shin (2004b), Argenziano (2008) and Morris, Shin, and Yildiz (2016)). Our results extends to the "common value" case easily. Indeed, in our model the "payoff type" is a linear function $x_{it} = \theta_{t-1} + \sigma z_{it}$ of θ_{t-1} and the normalized type z_{it} . In the "common value" case, the payoff type is a nonlinear function $X(\theta_{t-1}, z_{it}) = E[\theta_t | z_{it}, \theta_{t-1}]$ of the same variables with the same qualitative properties; indeed it approaches x_{it} in the limit cases $\sigma \rightarrow 0$ and $z_{it} \rightarrow \infty$. Our analyses and results extend to the common value case by use of the nonlinear function at expense of expositional clarity.*

3 Fat Tails and Rank Beliefs

"Rank beliefs" are key to our analysis: how does a player think his payoff parameter relates to others'? Formally, we define the *rank belief* of player i as the probability he assigns to the event that another player's signal x_j is lower than his own:

$$R(z) = \Pr(x_j \leq x_i | x_i = \theta_{-1} + \sigma z) = \frac{\int F(\varepsilon) f(\varepsilon) g(z - \varepsilon) d\varepsilon}{\int f(\varepsilon) g(z - \varepsilon) d\varepsilon}. \quad (7)$$

¹Morris, Shin, and Yildiz (2016) studied symmetric continuum player binary action games, and said that an action is Laplacian if it is a best response to uniform probability distribution over the proportion of other players taking that action. Our definition of latent corresponds to Laplacian in the particular game studied here. The latent equilibrium is then the one where all players are choosing the latent action. It corresponds to a many player version of risk dominance (Harsanyi and Selten (1988)) suitably adapted to the case of continuum player binary action games. We use the less specific term "latent" to suggest that the latent action would be meaningful in more general coordination games. For example, when we extend our model by dropping Assumption 1 in Section 7, the latency threshold becomes $\lim_{z \rightarrow \infty} R(z)$.

We refer to the function R as the *rank-belief function*. Note that rank belief is a function of normalized payoff z_i . Note that rank-beliefs depend only on the normalized payoff, with changes in θ_{-1} and changes in σ impacting rank-beliefs only through their impact on normalized payoffs. The next lemma states some useful properties of R .

Lemma 1 *The function R is differentiable and satisfies the following properties:*

Symmetry $R(-z) = 1 - R(z)$; in particular, $R(0) = 1/2$.

Single Crossing $R(z) > 1/2 > R(-z)$ whenever $z > 0$.

Uniform Limit Rank Beliefs $R(z) \rightarrow \frac{1}{2}$ as $z \rightarrow \infty$

That is, R is symmetric around $1/2$ for positive and negative values. It takes the value of $1/2$ at $z = 0$ and remains above $1/2$ for positive z , and symmetrically remains below $1/2$ for negative z . As $z \rightarrow \infty$, the rank belief converges back to $1/2$. Uniformity of limit rank beliefs implies immediately some further properties. The rank belief R is bounded away from 0 and 1. We write $\bar{R} < 1$ for the upper bound. And the rank belief is decreasing over some interval.

A leading example satisfying fat tails is where $g(\cdot)$ is the t -distribution and $f(\cdot)$ is the standard normal distribution. We plotted rank beliefs for this case in Figure 1 in the introduction. We use this example (unless otherwise noted) to illustrate our results throughout the paper.

We will make one last assumption directly on the rank beliefs function, requiring that σ is small:

Assumption 3 *There exists $z > 0$ such that $2\sigma z < R(z) - R(-z)$.*

A sufficient condition for this assumption is that $\sigma < R'(0)$. That is, the slope of the rank belief at a zero shock is larger than the slope of the payoff function with respect to the normalized shock. In this case, there will be multiple equilibrium actions for types near θ_{-1} . Morris and Shin (2004a) provides a cutoff between unique equilibrium and multiple equilibrium for some types when both common and idiosyncratic shocks are normally distributed. For normal distributions, this assumption states that we are in their multiplicity region. This assumption sharply distinguishes this work from mostly the existing literature

on global games: most of that literature focusses on situations when there is a globally unique equilibrium or on when there exist multiple equilibria. By contrast, this paper focusses on the case where there are multiple equilibria but is driven by the fact that—within those multiple equilibria—there are regions of fundamentals where a unique action is consistent with equilibrium. See Section 8 for further discussion.

We will maintain the Assumptions 1 through 3 throughout the paper without explicitly stating them. We will return in Section 7 to see what happens to rank beliefs and thus our results when we relax Assumption 1 (fat tails).

4 One-Shot Game

In the one-shot game, $G(\theta_{-1})$, the fundamental state is $\theta = \theta_{-1} + \sigma\eta$, where θ_{-1} is its prior mean and $\sigma\eta$ is a common shock. Each player i has a payoff type $x_i = \theta + \sigma\varepsilon_i$ and takes an action $a_i \in A$. Thus a strategy is a mapping $s_i : \mathbb{R} \rightarrow A$, where $s_i(x_i) \in A$ is the action of i if he is type x_i . A *Bayesian Nash equilibrium* (henceforth BNE) of the one-shot game $G(\theta_{-1})$ is defined as usual by requiring each type to play a best response. Note that $G(\theta_{-1})$ is a monotone supermodular game (Van Zandt and Vives (2007)), where the Bayesian Nash equilibria and the rationalizable strategies are bounded by monotone Bayesian Nash equilibria.

4.1 Characterization

We start with a full characterization of the structure of BNE. We first characterize a class of symmetric "threshold" equilibria. It will be convenient to work with players' normalized types $z_i = \frac{x_i - \theta_{-1}}{\sigma}$ introduced above. Suppose that each player invested only if his normalized type z_i were greater than a critical threshold \hat{z} . Consider a player whose normalized type was that critical threshold \hat{z} . His payoff to investing would be

$$\overbrace{\theta_{-1} + \sigma\hat{z}}^{\text{own payoff type}} + \overbrace{1 - R(\hat{z})}^{\text{expected proportion of others' investing}} - 1.$$

So that type will be indifferent only if this payoff is equal to 0, i.e.,

$$R(\hat{z}) = \theta_{-1} + \sigma\hat{z}. \tag{8}$$

This is thus a necessary condition for there to be a \hat{z} -threshold equilibrium. But this is also sufficient for equilibrium. Suppose that a player anticipated that all other players were going to play a \hat{z} -threshold strategy, and was therefore indifferent between investing and not investing when his normalized type was \hat{z} . If his normalized type was $z_i > \hat{z}$, he would have higher incentive to invest since both his payoff type would be higher and his expectation of the proportion of others' investing would be higher by Assumption 2.

The largest and smallest threshold strategy equilibria will play a key role in our analysis. Write $z^*(\theta_{-1})$ and $z^{**}(\theta_{-1})$ for the smallest and the largest solutions to (8), respectively; see Figure 2 for an illustration with our ongoing (t -distribution) example. We write $x^*(\theta_{-1}) = \sigma z^*(\theta_{-1}) + \theta_{-1}$ and $x^{**}(\theta_{-1}) = \sigma z^{**}(\theta_{-1}) + \theta_{-1}$ for the corresponding unnormalized types. In the one-shot game, all rationalizable strategies are bounded by monotone equilibria corresponding to $x^*(\theta_{-1})$ and $x^{**}(\theta_{-1})$: not invest is uniquely rationalizable whenever $x_i < x^*(\theta_{-1})$; invest is uniquely rationalizable whenever $x_i > x^{**}(\theta_{-1})$, and both actions are rationalizable for types between $x^*(\theta_{-1})$ and $x^{**}(\theta_{-1})$.

Hence, the Bayesian Nash equilibria are bounded by the corresponding extremal equilibria s^* and s^{**} , where

$$s_i^*(x_i) = \begin{cases} \text{Invest} & \text{if } x_i \geq x^*(\theta_{-1}) \\ \text{Not Invest} & \text{otherwise} \end{cases}$$

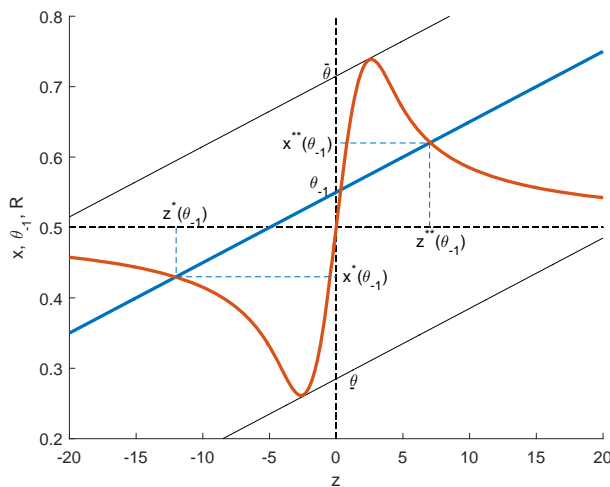
$$s_i^{**}(x_i) = \begin{cases} \text{Invest} & \text{if } x_i \geq x^{**}(\theta_{-1}) \\ \text{Not Invest} & \text{otherwise} \end{cases}$$

everywhere. Here, s^* is the equilibrium with the most investment, while s^{**} is the equilibrium with the least investment.

4.2 Majority Play and Shocks to Fundamentals

The above analysis characterized extremal equilibria, and thus rationalizable actions, for all players in a game. We now study instead *majority* behavior. There are two related reasons for doing this. First, it allows us to focus on a statistic of play that depends only on the fundamental state and not on idiosyncratic shocks. Second, when σ is small, nearly all players take the same action taken by majority.

Definition 1 *We say that an action a is majority uniquely rationalizable (henceforth MUR) at θ under θ_{-1} if action a is the only rationalizable action in game $G(\theta_{-1})$ for a majority of*

Figure 2: Extremal BNE cutoffs ($\sigma = 0.01$; $\theta_{-1} = 0.55$).

types when the fundamental value is θ .

In particular, invest is MUR at θ under θ_{-1} if and only if $\theta > x^{**}(\theta_{-1})$, and not invest is MUR if and only if $\theta < x^*(\theta_{-1})$. To see why, observe that an action is MUR only if it is uniquely rationalizable for the median player which is also the player who has an idiosyncratic shock of 0.

If an action is MUR, the analyst knows that the majority must take that action, independent of equilibrium selection. We will identify when there is a MUR action and how its existence depends on the level of fundamentals, θ , the prior mean θ_{-1} , and the size of the shock, $\theta - \theta_{-1}$, or difference in fundamentals from the prior mean.

Towards this goal, define the cutoff $\bar{\theta} \in (1/2, \bar{R})$ as the largest θ for which there exists $z > 0$ such that

$$R(z) \geq \sigma z + \theta. \quad (9)$$

As illustrated in Figure 2, the cutoff $\bar{\theta}$ is determined by the tangency of the line $\sigma z + \theta$ to R . Note that $\bar{\theta} < \bar{R} = \sup_z R(z)$ by definition, and $\bar{\theta} > 1/2$ by Assumption 3 and by $R(0) = 1/2$. Define also the cutoff $\underline{\theta} = 1 - \bar{\theta}$. We will refer to $\bar{\theta}$ and $\underline{\theta}$ as the *fundamental investment threshold* and *fundamental no investment threshold*, respectively. The fundamental investment threshold $\bar{\theta}$ is \bar{R} in the limit $\sigma \rightarrow 0$, and it decreases towards $1/2$ as σ increases.

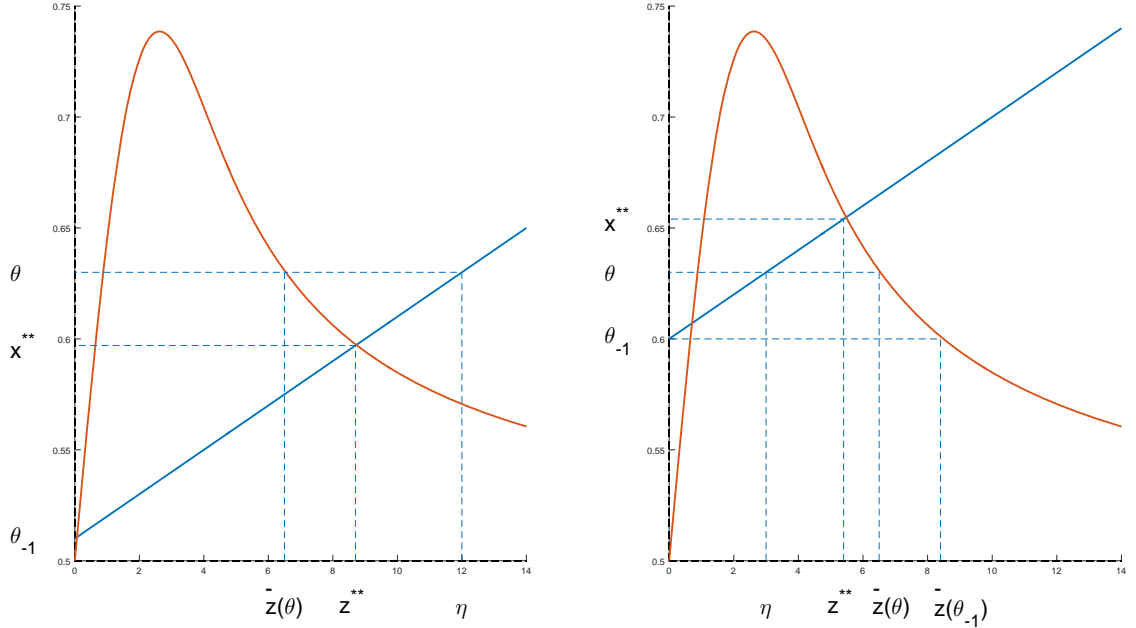


Figure 3: Effect of shocks; shock size is above the critical value on the left panel and below the critical value on the right panel.

For each $\theta > 1/2$, at which invest is latent, define also the cutoff

$$\bar{z}(\theta) = \max R^{-1}(\theta). \tag{10}$$

The cutoff $\bar{z}(\theta)$ is illustrated in Figure 3, where we only show the part of Figure 2 where invest is latent (i.e., $z \geq 0$ and $\theta \geq 1/2$). As seen in the figure, for $\theta \leq \bar{R}$, $\bar{z}(\theta)$ is the maximum level of shock z under which a player’s rank belief is θ . For $\theta > \bar{R}$, $\bar{z}(\theta) = -\infty$ by the convention that maximum of empty set is $-\infty$.

It turns out that the cutoff $\bar{z}(\theta)$ is the critical (normalized) threshold for a shock to be effective in making the latent action uniquely rationalizable for a majority. In the dynamic game, this will translate to equilibrium shifts. This is formally established in our next result—the main result of this section and the essence of our paper.

Proposition 1 *Invest is majority uniquely rationalizable if it is latent (i.e. $\theta > 1/2$) and*

$$\theta - \theta_{-1} > \sigma \bar{z}(\theta); \tag{11}$$

conversely, assuming R is single peaked on \mathbb{R}_+ and $\theta_{-1} \leq \bar{R} - \sigma \bar{z}(\bar{R})$, invest is not majority uniquely rationalizable if it is not latent or $\theta - \theta_{-1} \leq \sigma \bar{z}(\theta)$.

Proposition 1 identifies when invest is uniquely rationalizable for a majority: it is latent (i.e., $\theta > 1/2$) and there was a large positive shock with size more than critical level $\sigma \bar{z}(\theta)$. Moreover, under additional conditions,² the converse is also true: invest is not majority uniquely rationalizable if it is not latent or the shock size is smaller than the critical level. In that case, any action can be played by the majority in an equilibrium. By symmetry, this also establishes that not invest is majority uniquely rationalizable if and only if it is latent (i.e. $\theta < 1/2$) and there is a large negative shock—with size more than $\sigma \bar{z}(1 - \theta)$.

We will refer to $\sigma \bar{z}(\theta)$ as the *critical shock size*. The critical shock size is proportional to σ . Hence, it can be arbitrarily small for small σ . For example, in highly stable environments where one does not expect large shifts in fundamentals and payoffs, a very small positive jump in fundamentals will lead a majority to invest if invest is latent. Similarly, a very small drop in fundamentals will lead majority not to invest if not investing is latent. Note however that since the normalized critical shock size $\bar{z}(\theta)$ is independent of σ , the likelihood of such shifts is also independent of σ conditional on the current value of the fundamentals. In the remainder of the paper, by a "large shock", we mean a shock of size that exceeds a critical size proportional to σ .

The proof of Proposition 1 is as illustrated in Figure 3. To see why a large shock makes invest majority uniquely rationalizable when it is latent, consider the case depicted on the left panel. Here, $\theta = 0.63$ and hence invest is latent. The prior mean is $\theta_{-1} = 0.51$, and the difference exceeds the critical shock size: $\theta - \theta_{-1} > \sigma \bar{z}(\theta)$. That is, the normalized common shock $\eta = (\theta - \theta_{-1})/\sigma$ is larger than the cutoff $\bar{z}(\theta)$ —as in the figure. Now, for any shock level $z \geq \eta$, since z is strictly greater than $\bar{z}(\theta)$, the rank belief $R(z)$ is strictly below θ (by definition of $\bar{z}(\theta)$). But clearly for any such z , the unnormalized payoff type $\theta_{-1} + \sigma z$ is above θ . Hence, the payoff types remain strictly above the rank beliefs for all $z \geq \eta$. Therefore, the maximal equilibrium cutoff $z^{**}(\theta_{-1})$ is strictly smaller than η , and the associated payoff cutoff $x^{**}(\theta_{-1})$ is strictly below θ . Therefore, at θ , invest is uniquely rationalizable for all types with non-negative idiosyncratic shocks, who form a majority.

²The additional conditions in the converse are not superfluous. If $\theta_{-1} > \bar{\theta}$, by definition, $z^{**}(\theta_{-1}) < 0$, and hence $x^{**}(\theta_{-1}) < 1/2$. Therefore, invest is MUR but not latent for some θ . One can also construct multi-peaked rank belief functions under which invest may be MUR although (11) fails. The rank belief function is single-peaked in all our examples.

To see the converse, consider the case depicted in the right panel. Now, the fundamentals are as before ($\theta = 0.63$), but the prior mean is much higher: $\theta_{-1} = 0.6$. In particular, the normalized shock η is now smaller than the critical level $\bar{z}(\theta)$. The additional conditions for the converse are also met in this example: R is single-peaked, and $\theta_{-1} \leq \bar{R} - \sigma \bar{z}(\bar{R})$, so that R is decreasing at the cutoff $z^{**}(\theta_{-1})$, where the line $\theta_{-1} + \sigma z$ cuts R . Then, as in the figure, the equilibrium cutoff $z^{**}(\theta_{-1})$ must be at least as large as η , and thus $x^{**}(\theta_{-1})$ must be above θ .³ Consequently, not invest is rationalizable for types with negative idiosyncratic shocks, and therefore invest is not majority uniquely rationalizable. The proof in the appendix makes these arguments more formally.

Intuitively, in our ongoing example, the players are uncertain about the variance of underlying common shock, i.e., the underlying statistical model. When a player experiences an extremely large payoff shock, he updates his beliefs about the variance—his model—drastically and attributes an increasingly large portion of his payoff shock to a large common shock to the fundamentals. Hence, he believes nearly half of the players have a higher payoff than his own. When his payoff type is near the dominance trigger for investment, he then believes that invest is a dominant action for nearly half of the players, and invest is the only best response to such a belief-payoff pair. Invest is uniquely rationalizable for that type. Then, invest is also uniquely rationalizable for a player with a somewhat smaller but still large shock because she also assign substantial probability to those above him, who are investing. As we consider players with smaller shocks (and payoffs), the updating of the model becomes less drastic, leading to larger rank beliefs, and making above contagion weaker. Such a contagion of investment goes through so long as the payoff type remains above the rank belief. Our proof above shows that this is indeed the case if the shock size is above the critical level. Here, for the same level of payoff type, a player is more motivated to invest when the prior mean is lower—because he updates his model more drastically.⁴ In contrast, without variance uncertainty, he attributes a constant share of shocks to his own idiosyncratic shock. Hence, when the prior mean is lower, he would be less optimistic about

³Indeed, suppose $z^{**}(\theta_{-1}) < \bar{z}$. Then, since the straight line has positive slope, $x^{**}(\theta_{-1})$ would have been strictly below θ , and this would be a contradiction: R would be decreasing from $x^{**}(\theta_{-1})$ at $z^{**}(\theta_{-1})$ to the larger value θ at $\bar{z}(\theta)$.

⁴Fat-tailed distributions also lead to non-monotone beliefs in Acemoglu, Chernozhukov, and Werning (2016), where players are uncertain about the underlying statistical model. In that paper, this in turn leads to asymptotic belief disagreement.

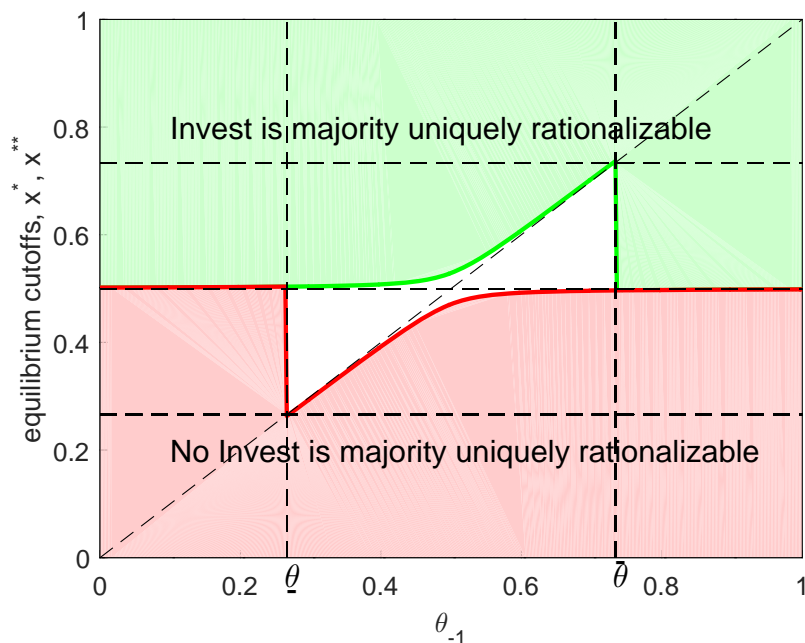


Figure 4: Equilibrium cutoffs and MUR regions

the other players and would be motivated to invest less. In that case, shocks do not play a role per se—as we will formally establish in Section 7. While this intuition can be made precise in our ongoing example, it provides intuition for our results more generally.

The qualitative properties of rationalizability in our model—many of which are implied by Proposition 1—are depicted in Figure 4, where we plot equilibrium cutoffs and regions in which invest and not invest are MUR actions for our ongoing example. When $\theta_{-1} \in [\underline{\theta}, \bar{\theta}]$, the upper equilibrium cutoff $x^{**}(\theta_{-1})$ is above $\max\{\theta_{-1}, 1/2\}$ and approaches $\max\{\theta_{-1}, 1/2\}$ as $\sigma \rightarrow 0$ by Proposition 1. Hence, when $\theta_{-1} > 1/2$, a large shock (relative to σ) makes invest majority uniquely rationalizable. Since $x^{**}(\theta_{-1}) > \theta_{-1}$, if the positive shock is sufficiently small, then neither action is MUR and both invest and not invest can be played by a majority in equilibrium. Likewise, the lower cutoff $x^*(\theta_{-1})$ is below $\min\{\theta_{-1}, 1/2\}$ and approaches $\min\{\theta_{-1}, 1/2\}$ as $\sigma \rightarrow 0$. Once again, large negative shocks make not invest MUR when $\theta_{-1} < 1/2$, while there is no MUR action under smaller shocks. Note that when $\theta_{-1} \in [\underline{\theta}, \bar{\theta}]$, MUR regions are confined to different sides of cutoff $\theta = 1/2$, and only a latent action can be MUR. Outside of $[\underline{\theta}, \bar{\theta}]$, a non-latent action can be MUR. For example, when $\theta_{-1} > \bar{\theta}$,

the cutoff $x^{**}(\theta_{-1})$ is slightly below $1/2$ —approaching $1/2$ by Proposition 1. Excluding such vanishingly small regions, latency is necessary for being MUR.

We next present some important implications of Proposition 1 for our paper. First, observe from Figure 4 that invest is MUR when θ is above a level, regardless of prior mean. Indeed, for any σ , the maximum value of $x^{**}(\theta_{-1})$ is \bar{R} , and \bar{R} provides a cutoff for such parameter-independent threshold. This is our "level" result:

Corollary 1 *Invest is majority uniquely rationalizable whenever $\theta > \bar{R}$; not invest is majority uniquely rationalizable whenever $\theta < 1 - \bar{R}$.*

Proof. When $\theta > \bar{R}$, invest is latent, and (11) holds vacuously because $\bar{z}(\theta) = -\infty$. ■

Recall that our fat-tail assumption implies that $\bar{R} < 1$. Hence, there remains a region $(0, 1 - \bar{R}) \cup (\bar{R}, 1)$ of fundamentals for which there is a uniquely rationalizable action for majority although no action is dominant. This is one point in our analysis where our maintained fat-tail assumption matters. We will examine what happens without fat tails in Section 7. The role of bounded rank beliefs is vividly illustrated in Figure 2. We have $\bar{R} \cong 0.735$, and hence there is a unique rationalizable action for majority in 53% of those cases without a dominant action—regardless of the parameters σ and θ_{-1} .

We next fix θ_{-1} and σ and determine the shock sizes that guarantee a MUR action. As we have observed in Figure 4, when $\theta_{-1} > 1/2$ and σ is small, any substantial positive shock makes invest MUR. Indeed, Proposition 1 implies that if an action is latent under θ_{-1} , a large shock with respect to σ and θ_{-1} will make it MUR:

Corollary 2 *Invest is majority uniquely rationalizable if it is latent under both current fundamentals (i.e. $\theta > 1/2$) and the prior mean (i.e. $\theta_{-1} > 1/2$) and*

$$\theta - \theta_{-1} > \sigma \bar{z}(\theta_{-1});$$

not invest is majority uniquely rationalizable if it is latent under both current fundamentals and the prior mean and

$$\theta - \theta_{-1} < -\sigma \bar{z}(1 - \theta_{-1}).$$

Proof. If $\theta_{-1} > \bar{\theta}$, $x^{**}(\theta_{-1}) < 1/2$, and hence invest is MUR whenever it is latent. Take any θ_{-1} with $\bar{\theta} \geq \theta_{-1} > 1/2$ and any θ with $\theta - \theta_{-1} > \sigma \bar{z}(\theta_{-1})$. Note that \bar{z} is non-increasing, and hence $\bar{z}(\theta_{-1}) \geq \bar{z}(\theta)$ —as in the right panel of Figure 3. Therefore,

$$\theta - \theta_{-1} > \sigma \bar{z}(\theta_{-1}) \geq \sigma \bar{z}(\theta),$$

showing by Proposition 1 that invest is MUR. ■

Corollary 2 provides a version of our "large shock" result in terms of prior means: a latent action is uniquely rationalizable for a majority if it was latent under prior mean and the shock size exceeds the critical level with respect to the prior mean. For example, if $\theta_{-1} > 1/2$, a positive shock larger than the critical size $\sigma \bar{z}(\theta_{-1})$ makes invest uniquely rationalizable for a majority. The attraction of this result is that the critical shock size does not depend on the realization of the fundamentals. In particular, for $\theta_{-1} \in (1/2, \bar{R}]$, it yields a lower bound on probability of investment becoming MUR:

$$\Pr(\theta > x^{**}(\theta_{-1})) \geq \Pr(z > \bar{z}(\theta_{-1})) = 1 - G(\bar{z}(\theta_{-1})).$$

Observe that $\bar{z}(\theta_{-1})$ is decreasing in θ_{-1} , and hence the lower bound $1 - G(\bar{z}(\theta_{-1}))$ for investment becoming MUR is increasing in θ_{-1} .⁵ Near $\theta_{-1} = 1/2$, it is nearly 0, and as θ_{-1} increases to \bar{R} , it increases to $1 - G(\bar{z}(\bar{R}))$, which is less than 1/2. For $\theta_{-1} > \bar{R}$, it yields the lower bound $1 - G((1/2 - \theta_{-1})/\sigma)$, which is greater than 1/2 and approaches 1 as $\sigma \rightarrow 0$. The lower bound is not tight, but it becomes tight as $\sigma \rightarrow 0$.

What if invest was not latent under prior mean (i.e., $\theta_{-1} \leq 1/2$)? In that case, Proposition 1 implies that it is unlikely that invest is MUR. Since $\theta_{-1} < \bar{\theta}$, invest can be MUR only if invest is latent, i.e., $\theta > 1/2$. This requires a shock of size more than $1/2 - \theta_{-1}$, where the normalized shock is of order $1/\sigma$. The probability of such a shock goes to zero as $\sigma \rightarrow 0$. In our discussions, we call normalized shocks of order $1/\sigma$ *very large*. Proposition 1 further implies that a very large shock that makes an action latent also makes it MUR.

Corollary 3 *For any $\theta_{-1} < 1/2$ and any $\theta > 1/2$, there exists $\bar{\sigma}$ such that invest is MUR at θ under θ_{-1} for all $\sigma \in (0, \bar{\sigma})$.*

Proof. Take $\bar{\sigma} = (\theta - \theta_{-1})/\bar{z}(\theta)$. Then, by Proposition 1, for any $\sigma < \bar{\sigma}$, we have $\theta - \theta_{-1} > \sigma \bar{z}(\theta)$, and hence invest is MUR. ■

We next focus on the case that the shock size $|\theta - \theta_{-1}|$ is small in absolute terms, i.e., $\sigma\eta$ is small. In particular, note that by continuity of x^{**} on $[\underline{\theta}, \bar{\theta}]$, we have $x^{**}(\theta_{-1}) - \theta_{-1} \geq \Delta$ for some $\Delta \in (0, \bar{\theta} - 1/2]$. When the shock size is smaller than Δ , there is no MUR action for $\theta_{-1} \in [\underline{\theta}, \bar{\theta}]$. This is vividly illustrated in Figure 4: for $\theta_{-1} \in [\underline{\theta}, \bar{\theta}]$, the unshaded region

⁵The probability of invest becoming MUR is also weakly increasing in θ_{-1} (by Lemma 3 in Section 7 below).

strictly contains the region between $\max\{\theta_{-1}, 1/2\}$ and $\min\{\theta_{-1}, 1/2\}$. In contrast, as in the figure, the latent action is MUR for θ_{-1} outside $[\underline{\theta}, \bar{\theta}]$. We formally state these next (and prove in the appendix).

Proposition 2 *There exists $\Delta > 0$ such that whenever $|\theta - \theta_{-1}| \leq \Delta$, invest is majority uniquely rationalizable if and only if $\theta_{-1} > \bar{\theta}$ and not invest is majority uniquely rationalizable if and only if $\theta_{-1} < \underline{\theta}$.*

The bound Δ may depend on the rank belief function, and thus on F , G and σ , but is independent of θ_{-1} . Proposition 2 establishes that, without a large shock, investment is uniquely rationalizable for a majority when $\theta_{-1} > \bar{\theta}$ and no investment is uniquely rationalizable for a majority of players when $\theta_{-1} < \underline{\theta}$. But equilibrium play will depend on equilibrium selection when θ_{-1} is in the intermediate range $[\underline{\theta}, \bar{\theta}]$, as the action choice depends on which equilibrium is played. The focus of our analysis will be inside this range where fundamentals alone do not determine if there is a MUR action and what it is. Here, Δ is given by the difference $x^{**}(\bar{\theta}) - \bar{\theta}$. As illustrated in Figure 4, in the limit $\sigma \rightarrow 0$, $x^{**}(\bar{\theta})$ approaches $\bar{\theta}$ and the cutoff Δ for a shock to be small also goes to 0. Note that the range $(-\infty, 1 - \bar{\theta}) \cup (\bar{\theta}, \infty)$ of prior means for which there is a MUR action is increasing in the size of uncertainty σ . It starts with $(-\infty, 1 - \bar{R}) \cup (\bar{R}, \infty)$ when σ is close to zero and monotonically grows and approaches $\mathbb{R} \setminus \{1/2\}$ as σ grows.

The results in this section can be used to give a heuristic explanation of the dynamic analysis that follows. We have investment played by a majority if invest is latent and there is a large positive shock—as in Proposition 1. If a majority not invested in the previous period, then the shock leads to a shift in the majority behavior towards investment. This is true for all equilibria. In some equilibria, equilibrium behavior may shift even without a shock because the equilibrium behavior can be conditioned on the history somewhat arbitrarily. If one rules out such spurious shifts by imposing inertia, then small changes lead to equilibrium shifts only when the fundamental moves across the boundaries $\underline{\theta}$ and $\bar{\theta}$, when a particular action becomes MUR in the one-shot game—as in Proposition 2.

5 Dynamic Model—Equilibrium Shifts and Shocks

In this section, turning back to the dynamic model, we identify equilibrium shifts and the role the shocks play in such shifts. We formally describe strategies in the dynamic game and the

relevant solution concept of public perfect Bayesian equilibrium (PPBNE). We show that, in any PPBNE, the majority takes the latent action if there is a large shock—proportional to σ —in the direction of that action. A positive large shock leads to majority investing if $\theta_t > 1/2$, and a negative large shock leads to majority not investing if $\theta_t < 1/2$. Such shocks lead to equilibrium shifts if the majority did not play the latent action in the previous round. Without a large shock, in equilibrium, majority of players must invest if the previous fundamental exceeds $\bar{\theta}$, and must not invest if it goes below $\underline{\theta}$, leading to possible equilibrium shifts as the fundamental crosses the above thresholds.

5.1 Solution Concept

A *public history* at a period t is a sequence $h_t = (\theta_{-1}, \theta_0, \theta_1, \dots, \theta_{t-1}; \alpha_0, \alpha_1, \dots, \alpha_{t-1})$. Each player also has a *private history* $h_{t,i} = (h_t; x_{i0}, \dots, x_{i,t-1}; a_{i0}, \dots, a_{i,t-1})$. A *strategy* of a player i is a mapping s_i that maps each private history $h_{t,i}$ and signal x_{it} to an action $s_i(h_{t,i}, x_{it}) \in A$ for each t . A strategy s_i is *public* if $s_i(h_{t,i}, x_{it})$ depends only on h_t and x_{it} , and we will simply write as $s_i(h_t, x_{it})$. We will write s^{**} and s^* for the extremal equilibria of the one-shot game. A *public perfect Bayesian equilibrium* (henceforth PPBNE) is public strategy profile s in which $s_i(h_t, x_{it})$ is a conditional best response to s_{-i} for each $(i, t, h_{t,i}, x_{it})$. As noted above, for any θ_{t-1} , the analysis reduces to the analysis of the one-shot Bayesian game $G(\theta_{t-1})$ described in the previous section:

Lemma 2 *A public strategy profile s is PPBNE if and only if $s(h_t, \cdot)$ is a BNE of $G(\theta_{t-1})$ for each $h_t = (\theta_{-1}, \dots, \theta_{t-1}; \alpha_0, \dots, \alpha_{t-1})$.*

Thus there is a best PPBNE where the best static equilibrium s^* is played at every history. There is a worst PPBNE where the worst static equilibrium s^{**} is played at every history. Or the dependence on history may be arbitrary. Our main result applies to all PPBNE, showing that a large shock leads a majority to play the latent action, as we formally establish next.

5.2 Equilibrium Shifts

We start with a few definitions. Consider any PPBNE s . At any given history h_t , the aggregate investment under s is

$$\alpha(h_t, \theta_t | s) = \int_{\{\varepsilon: s(h_t, \theta_t + \sigma\varepsilon) = \text{invest}\}} f(\varepsilon) d\varepsilon,$$

the fraction of players who invest—as a function of history and θ_t . We say that *there is majority investment* at (h_t, θ_t) under equilibrium s if $\alpha(h_t, \theta_t | s) > 1/2$. Similarly, we say that *there is minority investment* at (h_t, θ_t) under equilibrium s if $\alpha(h_t, \theta_t | s) < 1/2$. We also say that there is an *equilibrium shift* at history (h_t, θ_t) if there is minority investment at h_{t-1} and a majority investment at (h_t, θ_t) or vice versa.

As a corollary to Proposition 2, the next result states that a positive large shock leads to majority investment if invest is latent, resulting in an equilibrium shift if a majority did not invest in the previous period.

Proposition 3 *Under any PPBNE s , at any history h_t , there is majority investment at (h_t, θ_t) if invest is latent (i.e. $\theta_t > 1/2$) and*

$$\theta_t - \theta_{t-1} > \sigma \bar{z}(\theta_t);$$

there is minority investment at (h_t, θ_t) if not invest is latent (i.e. $\theta_t < 1/2$) and

$$\theta_t - \theta_{t-1} < -\sigma \bar{z}(1 - \theta_t).$$

Proof. By Lemma 2, $s(h_t, \cdot)$ is a BNE of $G(\theta_{t-1})$. Hence, under $s(h_t, \cdot)$, a majority of players invest, leading to majority investment, if invest is majority uniquely rationalizable at θ_t under θ_{t-1} . But Proposition 1 has established that this is indeed the case if $\theta_t > 1/2$ and $\theta_t - \theta_{t-1} > \sigma \bar{z}(\theta_t)$, proving the first half. Second part is by symmetry. ■

Proposition 3 establishes that a majority of players take the latent action whenever there is a large shock towards the latent action. For example, if invest is latent and there was a large positive shock with size larger than $\sigma \bar{z}(\theta_t)$, then there is majority investment at (h_t, θ_t) . If there was minority investment at h_t , then this results in an equilibrium shift. Such an equilibrium shift can happen in two distinct scenarios:⁶

1. invest was latent at $t - 1$ and a *large* positive shock leads majority to switch investing (as in Corollary 2), or
2. not invest was latent at $t - 1$ and a *very large* positive shock makes invest latent and leads majority to switch investing (as in Corollary 3).

⁶There is also majority investment if $\theta_t > \bar{R}$ (as in Corollary 1), even if there was a negative shock. But there is no equilibrium shift in the latter case because there was also majority investment at $t - 1$.

Recall that if not invest was not latent, a large but not very large positive shock does not lead to an equilibrium shift. In that case, a negative shock with size larger than $\sigma \bar{z} (1 - \theta_{t-1})$ leads to minority investment. Once again the required size is proportional to σ , so that even a small shock to the fundamental may lead to an equilibrium shift when σ is small.

What if the size of the shock is small relative to σ ? In that case, for arbitrary equilibria, we can determine whether there is majority or minority investment according to whether θ_{t-1} is above $\bar{\theta}$ or below $\underline{\theta}$, using Proposition 2. Whether there is majority or minority investment in the intermediate region depends on the equilibrium played. This is formalized by our next result, which is a corollary to Proposition 2 (see the appendix for a proof).

Proposition 4 *There exists $\Delta > 0$ such that, under any PPBNE s , there is majority investment at any (h_t, θ_t) with $|\theta_t - \theta_{t-1}| \leq \Delta$ and $\theta_{t-1} > \bar{\theta}$, and there is minority investment at any (h_t, θ_t) with $|\theta_t - \theta_{t-1}| \leq \Delta$ and $\theta_{t-1} < \underline{\theta}$. Moreover, at any (h_t, θ_t) with $|\theta_t - \theta_{t-1}| < \Delta$ and $\theta_{t-1} \in [\underline{\theta}, \bar{\theta}]$, there is majority investment under s^* and minority investment under s^{**} .*

That is, without a large shock relative to σ , under any equilibrium, there is majority investment if the previous fundamental θ_{t-1} is above the cutoff $\bar{\theta}$, and there is minority investment if θ_{t-1} is below the cutoff $\underline{\theta}$. If the previous fundamental is in the intermediate range, then whether there is majority or minority investment depends on the equilibrium; in particular, there is majority investment under the most efficient equilibrium s^* , and there is minority investment in under the least efficient equilibrium s^{**} .

Now imagine that the fundamental drifts on a "continuous path" without a large shock. If the fundamental drifts above $\bar{\theta}$ while only minority are investing, there will be an equilibrium shift towards majority investment as it crosses the cutoff, and there will not be another equilibrium shift until the fundamental drifts back below $\bar{\theta}$. Likewise, if fundamental drifts below the cutoff $\underline{\theta}$ while majority are investing, there will be equilibrium shift at the cutoff—and there will not be any other shift while the fundamental remains below $\underline{\theta}$. If the fundamental remains within the intermediate range $[\underline{\theta}, \bar{\theta}]$, there will not be any equilibrium shift under the extremal equilibria, but since an equilibrium can condition on history arbitrarily by selecting any of the extremal equilibria within this region, there can be arbitrary equilibrium shifts within this regions. In order to avoid such spurious equilibrium shifts, we next focus on an equilibrium with hysteresis, in which the majority behavior does not change unless it becomes unsustainable as an equilibrium behavior.

6 Equilibrium Shifts under Hysteresis

In this section, we focus on a particular PPBNE, where players chose one of the (symmetric) extremal static equilibria s^* and s^{**} , playing the good investment equilibrium if a majority of players invested in the previous period and the bad no investment equilibrium if a majority of players did not invest in the previous period.⁷

We will also assume that players start out in the "good equilibrium". Thus we will define the hysteresis equilibrium \hat{s} by

$$\hat{s}(\emptyset, \cdot) = s^*(\cdot)$$

and

$$\hat{s}(h_t, \cdot) = \begin{cases} s^*(h_t, \cdot) & \text{if } \alpha_{t-1} \geq 1/2 \\ s^{**}(h_t, \cdot) & \text{if } \alpha_{t-1} < 1/2 \end{cases} \quad (t > 0).$$

Because of the hysteresis, equilibrium does not shift unless there is a large shock or the fundamental goes outside of the region $[\underline{\theta}, \bar{\theta}]$. This allows us a simple characterization of when equilibrium shifts occur, as stated in the next result. Under \hat{s} , if $\alpha_{t-1} < 1/2$, there is a shift to majority investment at (h_t, θ_t) if and only if

$$\theta_t > x^{**}(\theta_{t-1}).$$

Likewise, if $\alpha_{t-1} > 1/2$, then there is a shift to minority investment at (h_t, θ_t) if and only if

$$\theta_t < x^*(\theta_{t-1}).$$

Combined with this simple characterization, our previous results lead to the following description of equilibrium shifts under hysteresis.

Proposition 5 *Under \hat{s} , for any history h_t with minority investment (i.e., $\alpha < 1/2$), there is equilibrium shift at (h_t, θ_t) whenever*

1. *invest is latent and $\theta_{t-1} > \bar{\theta}$, or*

⁷Hysteresis as a selection device is often assumed as a modelling device, see Krugman (1991) and Cooper (1994) among others. Romero (2015) has tested hysteresis in the laboratory, confirming its existence in a setting with evolving complete information payoffs. The switches occur before dominance regions are reached, consistent with our results.

2. *invest is latent and $\theta_t - \theta_{t-1} > \sigma \bar{z}(\theta_t)$.*

Conversely, for $\theta_{t-1} \leq \bar{\theta}$, there is no equilibrium shift if invest is not latent or $|\theta_t - \theta_{t-1}| \leq \Delta$, where Δ is as in Proposition 2, and for $\theta_{t-1} \leq \bar{R} - \sigma \bar{z}(\bar{R})$, there is no equilibrium shift if $\theta_t - \theta_{t-1} \leq \sigma \bar{z}(\theta_t)$ and R is single peaked on \mathbb{R}_+ .

Under hysteresis, Proposition 5 provides nearly a characterization of when equilibrium shifts to majority investment occur: invest is latent and either we were above the fundamental investment threshold at $t - 1$ or there was a large positive shock at t . The converse rules out an equilibrium shift for all but the following two remaining cases, both are negligible when σ is small. First case is $\theta_{t-1} > \bar{\theta}$ but invest is not latent. This can happen when there is a very large negative shock. In that case, when σ is small, there is no equilibrium shift because there must be minority investment by Corollary 3; when σ is large, there could be an equilibrium shift despite invest not being latent when $\theta_t \in (x^{**}(\theta_t), 1/2)$. The second excluded case is θ_{t-1} is in $(\bar{R} - \sigma \bar{z}(\bar{R}), \bar{\theta}]$ and θ_t is in $(\Delta, \sigma \bar{z}(\theta_t)]$, where the lengths of both intervals are of order σ . In that case, equilibrium may shift to majority investment.

The dynamics under the hysteresis equilibrium \hat{s} is illustrated in Figure 5. Time is on the x axis. A sample path of fundamentals (in blue) and aggregate investment (in red) are plotted on the y axis. But aggregate investment is always close to one or close to zero, so majority and minority investment corresponds to almost all investing and almost all not investing, respectively. There are four periods of majority investment with varying lengths—the areas shaded (in green) between 0 and 1—interspersed with minority investment. At the beginning, investment is latent, and there is majority investment with aggregate investment nearly 1. As the fundamental drifts below $1/2$, the majority keeps investing due to hysteresis, although investing ceases to be latent. The arrival of negative shock shifts the equilibrium: the majority stop investing and aggregate investment drops near zero. This is the end of first period of majority investment. After that the fundamental drifts up, but aggregate investment remains near zero until the fundamental reaches the cutoff $\bar{\theta}$. The majority starts investing as the fundamental crosses $\bar{\theta}$. Sometime after the second period of majority investment starts, a large negative shock arrives, dropping the fundamental near $1/2$, but has no discernible effect on aggregate investment because investment remains latent. Later a smaller large shock shifts the equilibrium and ends the second period of majority investment, as it arrives when not investing is latent. This starts a long period of minority investment, where shocks do not shift the equilibrium because fundamental remains mainly

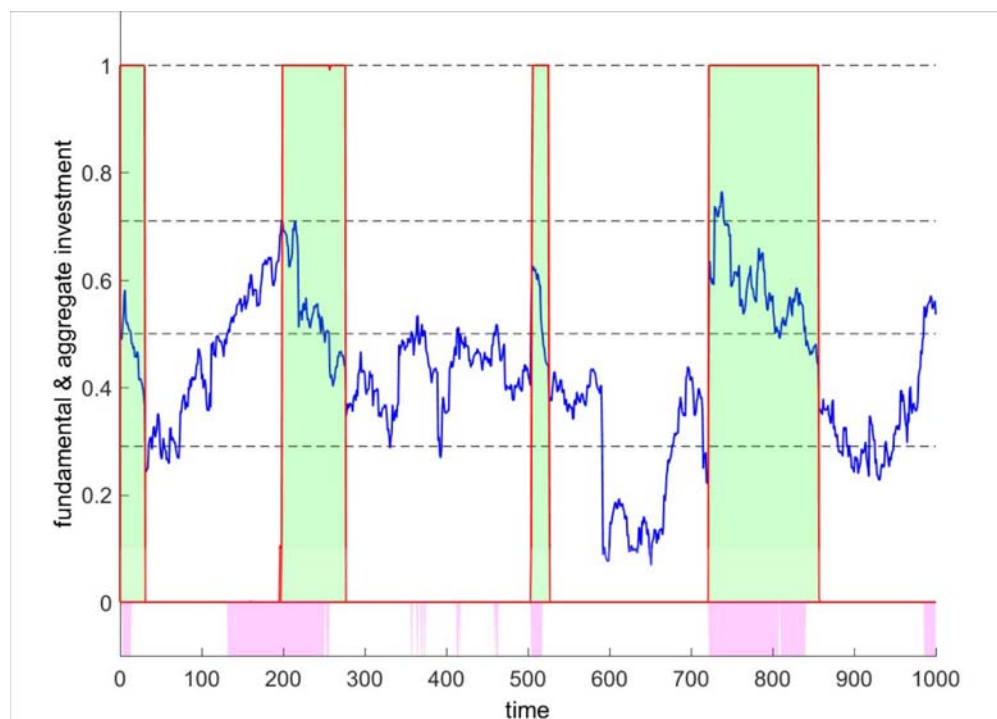


Figure 5: Equilibrium shifts on a typical sample path. The fundamental varies stochastically with occasional large shocks (in blue lines), while aggregate investment mostly remains either around 1 or around 0 with occasional switching (in red lines). The area under the aggregate investment is shaded (in green).

below $1/2$. Finally a very large positive shock arrives and makes the investment latent, shifting the equilibrium to majority investment. Next, the fundamental decreases sharply without causing an equilibrium shift because the drops come in different times, so there is no large shock. A relatively small large negative shock cuts short this spell of majority investment, as it arrives when not investing is latent. Shocks lead to two more equilibrium shifts in a similar fashion.

It is useful to compare the dynamics here to two usual solution concepts. First, consider the hysteresis equilibrium under complete information (as in Cooper (1994)) where (all) players switch their action only when the previous action becomes inconsistent with equilibrium, switching to all invest when θ goes above 1 and switching to nobody investing when θ goes below 0. Under this equilibrium, the players keep investing throughout because the fundamental never drops below 0. There are more equilibrium shifts in our model in general

because equilibrium shifts even before the fundamental reaches the cutoffs 0 and 1. Second, consider the classic global games solution in which all players play the risk dominant action. In the figure we shaded the area between -0.1 and 0 (in magenta) when all invest under this solution. The equilibrium shifts as the fundamental crosses $1/2$, resulting in frequent equilibrium shifts when the fundamental is near $1/2$ and no shift away from the cutoff. Our equilibrium is not sensitive the cutoff $1/2$ per se, but the outcomes correlate because shocks revert to the solution under risk dominance if they happen to be in the right direction.

7 Rank Beliefs Revisited

We now consider the implications of dropping the fat-tail assumption. Recall that the fat-tail assumption was used in establishing limit uniformity of rank beliefs, i.e., the property that $R(z) \rightarrow \frac{1}{2}$ as $z \rightarrow \infty$. In fact, limit uniformity was the only property of rank beliefs we used in our analysis (i.e., our results hold even without fat tails if limit uniformity continues to hold). But limit uniformity does not hold in general. Consider rank beliefs if both idiosyncratic and common shocks are normally distributed: these are plotted in Figure 6. The rank belief function is monotone and approaches 1, so that $\bar{R} = \sup_z R(z) = 1$. In the same figure, we also plot the case the common shocks are (double) exponentially distributed instead. Now the common shocks have thicker tails than the idiosyncratic shocks without having a fat tail. The rank belief function is still monotone but it is bounded away from 1 by $\bar{R} \cong 0.75$. This motivates the following definitions.

Definition 2 Rank beliefs are monotone if $R(z)$ is increasing in z .

Definition 3 Rank beliefs are bounded if $\bar{R} = \max_z R(z) < 1$.

Definition 4 Rank beliefs are limit certain if $\bar{R} = \lim_{z \rightarrow \infty} R(z) = 1$.

Note that rank beliefs are monotone in both exponential and normal examples; they are bounded under exponential distribution while limit certain under normal distribution. The following comparative statics about the extremal equilibria will be useful in our analyses for monotone rank beliefs.

Lemma 3 Extremal cutoff functions z^* and z^{**} are decreasing in θ_{-1} . Under monotone rank beliefs, x^* and x^{**} are also decreasing in θ_{-1} .

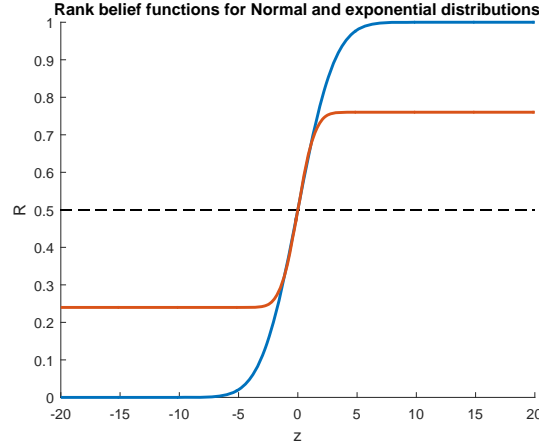


Figure 6: Rank belief function under normal idiosyncratic shocks and normal or exponential common shocks

The first part of the lemma is broadly true. As in Figure 2, at the extremal cutoffs, the line $\sigma z + \theta_{-1}$ cuts $R(z)$ from below. As θ_{-1} increases, $\sigma z + \theta_{-1}$ moves up, while $R(z)$ remains where it is. Consequently, the cutoffs z^* and z^{**} move to the left. The second part immediately follows from the first part: since $x^* = R(z^*)$ and $x^{**} = R(z^{**})$, the cutoffs x^* and x^{**} inherit the ordinal properties of z^* and z^{**} when R is monotone.

When rank beliefs are monotone, Lemma 3 implies that shocks per se do not lead to equilibrium shifts, and they are irrelevant in that the aggregate investment is determined by the levels, as stated next. (We write $\theta_{t-1}(h_t)$ for θ_{t-1} in history h_t .)

Proposition 6 *Under Assumptions 2-3 and monotone rank beliefs,*

1. *if invest is majority uniquely rationalizable at θ under θ_{-1} , then invest is majority uniquely rationalizable at θ under any θ'_{-1} with $\theta - \theta_{-1} \leq \theta - \theta'_{-1}$;*
2. *under the hysteresis equilibrium \hat{s} , for any history h_{t-1} with minority investment and for any θ_{t-1} and θ_t with $|\theta_t - \theta_{t-1}| < \Delta$, there is majority investment at history $(h_t, \theta_t) = (h_{t-1}, \theta_{t-1}, \alpha(h_{t-1}, \theta_{t-1} | \hat{s}), \theta_t)$ if and only if $\theta_{t-1} > \bar{\theta}$, where Δ is as in Proposition 2.*

Proof. Part 1 immediately follows from Lemma 3 and the fact that invest is MUR if and only if $\theta > x^{**}(\theta_{-1})$. To prove part 2, assume $\theta_{t-1} > \bar{\theta}$. Then, there is majority

investment at h_t by Proposition 4. Conversely assume that $\theta_{t-1} \leq \bar{\theta}$. Then, by Lemma 3, $\theta_{t-1} < x^{**}(\bar{\theta}) \leq x^{**}(\theta_{t-1})$, and hence by definition of \hat{s} , we have $\alpha(h_{t-1}, \theta_{t-1} | \hat{s}) < 1/2$. Therefore, since \hat{s} uses the cutoff x^{**} at h_t ,

$$\begin{aligned} \alpha(h_t, \theta_t | \hat{s}) &= 1 - F((\theta_t - x^{**}(\theta_{t-1})) / \sigma) \\ &< 1 - F((\Delta + \theta_{t-1} - x^{**}(\theta_{t-1})) / \sigma), \text{ by } |\theta_t - \theta_{t-1}| < \Delta, \\ &\leq 1 - F((\Delta + \bar{\theta} - x^{**}(\bar{\theta})) / \sigma), \text{ by Lemma 3,} \\ &\leq 1/2, \text{ by } \Delta \leq x^{**}(\bar{\theta}) - \bar{\theta}. \end{aligned}$$

■

Proposition 6 establishes that the shocks do not matter under monotone rank beliefs, as in the case of normal and exponential distributions. First, as stated in part 1, if a positive shock makes investment uniquely rationalizable, then invest would have been uniquely rationalizable even with a positive smaller shock that brings the fundamental to the same level (starting from a higher level of fundamentals the previous period). Second, as stated in part 2, excluding for two consecutive shocks, whether there is an equilibrium shift under \hat{s} is determined by the current level of fundamentals alone.

Proposition 6 further implies that, in the limit $\sigma \rightarrow 0$, the equilibrium shifts occur when the fundamental crosses across the levels \bar{R} and $1 - \bar{R}$.

Corollary 4 *Under Assumptions 2-3 and monotone rank beliefs,*

$$\lim_{\sigma \rightarrow 0} x^*(\theta_{-1}) = \begin{cases} 1 - \bar{R} & \text{if } \theta_{-1} > 1 - \bar{R} \\ \bar{R} & \text{otherwise} \end{cases} \quad \text{and} \quad \lim_{\sigma \rightarrow 0} x^{**}(\theta_{-1}) = \begin{cases} \bar{R} & \text{if } \theta_{-1} < \bar{R} \\ 1 - \bar{R} & \text{otherwise.} \end{cases}$$

Hence, under equilibrium \hat{s} , in the limit $\sigma \rightarrow 0$, there is equilibrium shift from minority investment to majority investment at (h_t, θ_t) only if $\theta_t \geq \bar{R}$, and there is equilibrium shift from majority investment to minority investment at (h_t, θ_t) only if $\theta_t \leq 1 - \bar{R}$.

Proof. By Proposition 6, for any $\theta_{-1} \leq \bar{\theta}(\sigma)$, we have $x^{**}(\bar{\theta}(\sigma)) \leq x^{**}(\theta_{-1}) < \bar{R}$. As $\sigma \rightarrow 0$, since $x^{**}(\bar{\theta}(\sigma)) \rightarrow \bar{R}$, this shows that $x^{**}(\theta_{-1}) \rightarrow \bar{R}$ for any $\theta_{-1} < \bar{R}$ (where the convergence is uniform on intervals $(-\infty, \theta_{-1})$). The other limits are obtained similarly by symmetry. If there is an equilibrium shift from minority investment to majority investment, we have $\theta_{t-1} < \bar{R}$ and $\theta_t > x^{**}(\theta_{t-1})$, where $x^{**}(\theta_{t-1}) \rightarrow \bar{R}$ by definition. ■

Specifically, if the rank beliefs are monotone and limit certain, then the equilibrium cutoffs converge to the dominance triggers, so that the equilibrium shifts only when the fundamental

enters a region in which an action is dominant, as in the complete information case. Hence, under normal distributions, the limit behavior is identical to the complete information case. It turns out that monotonicity of rank beliefs is dispensable for this result.

Corollary 5 *Under Assumptions 2-3 and limit certain rank beliefs,*

$$\lim_{\sigma \rightarrow 0} x^*(\theta_{-1}) = \begin{cases} 0 & \text{if } \theta_{-1} > 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \lim_{\sigma \rightarrow 0} x^{**}(\theta_{-1}) = \begin{cases} 1 & \text{if } \theta_{-1} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, under equilibrium \hat{s} , in the limit $\sigma \rightarrow 0$, there is equilibrium shift from minority investment to majority investment at (h_t, θ_t) only if $\theta_t \geq 1$, and there is equilibrium shift from majority investment to minority investment at (h_t, θ_t) only if $\theta_t \leq 0$.

We have established that shocks lead to equilibrium shifts under uniform limit rank beliefs, while they do not matter per se under monotone rank beliefs. The next result generalizes our characterization of MUR action (Proposition 1) to arbitrary rank belief functions, identifying when shocks lead to equilibrium shifts in general. (Here, the cutoff $\bar{z}(\theta) = \max R^{-1}(\theta_{-1})$ is defined for $\theta > R_\infty$.)

Proposition 7 *Under Assumptions 2-3, assume that*

$$R_\infty \equiv \lim_{z \rightarrow \infty} R(z)$$

exists. Then, invest is majority uniquely rationalizable if $\theta > R_\infty$ and

$$\theta - \theta_{-1} > \sigma \bar{z}(\theta_{-1}).$$

Conversely, assuming R is single peaked on \mathbb{R}_+ with $R_\infty < \bar{R}$ and $\theta_{-1} \leq \bar{R} - \sigma \bar{z}(\bar{R})$, invest is not majority uniquely rationalizable if $\theta < R_\infty$ or $\theta - \theta_{-1} \leq \sigma \bar{z}(\theta)$.

Proposition 7 extends Proposition 1 to arbitrary rank beliefs by simply setting the latency threshold as R_∞ ; one simply replaces $1/2$ with R_∞ in the proof of Proposition 1 to prove this result. Using Proposition 7 in place of Proposition 1, one can easily extend our qualitative results to arbitrary rank beliefs with the new latency threshold. Note that, under general rank belief functions, there is a middle region $[1 - R_\infty, R_\infty]$ on which no action is latent and large shocks do not play a role. Under uniform limit rank beliefs, this is the degenerate case of $\theta = 1/2$. Under limit certain rank beliefs, this is the entire region of multiplicity. Note also that under monotone rank beliefs (possibly without limit certainty), Proposition 7 reduces to the level result in Corollary 1: invest is MUR if $\theta > \bar{R}$ and not invest is MUR if $\theta < 1 - \bar{R}$.

8 Discussion

Economic models often give rise to multiple equilibria. What is the empirical counterpart of such models? There are two versions of this question. In a static setting, how can we explain which equilibrium is played? In a dynamic setting, how can we explain switches among equilibria?

One response to the static question is to observe that the multiplicity may be an artifact of the assumption of complete information, or common certainty of the game's payoffs. A first generation of *global game* models (Carlsson and Van Damme (1993), Morris and Shin (1998) and Morris and Shin (2003)) argued that if the common certainty assumption were relaxed a natural way, there is a unique equilibrium selection. Morris, Shin, and Yildiz (2016) formalized the idea that uniqueness and equilibrium selection that arose relied on the assumption of (common certainty of) uniform rank beliefs.⁸ The latent equilibrium is that unique selection. Note that in this literature, the focus is on *global* uniqueness: there is a unique prediction of play for any signal that a player might observe.

Two basic criticisms of this first generation of global models are the following. First, with respect to assumptions, uniform rank beliefs will not hold when there is approximate common certainty of payoffs, which is a reasonable assumption in many environments (for example, this arises when there are very accurate public signals).⁹ Second, with respect to predictions, as long as rank beliefs are approximately uniform, outcomes will be largely determined by fundamentals. Thus in a dynamic model, the prediction would be that equilibrium play would always be switching when fundamentals crossed a threshold (which we call the latency threshold). Both predictions are counter-factual.

In this paper, we made an intermediate set of assumptions, relative to complete information and first generation global games. Like the first generation global games literature, we relax complete information and use the vital insight that properties of rank beliefs sometimes lead to unique predictions.¹⁰ Like the complete information literature, we allowed from the

⁸Weinstein and Yildiz (2007) pointed out that it mattered exactly how common certainty assumptions were relaxed: any rationalizable action in the underlying complete information game is a uniquely rationalizable action for a type of a player that is "close" the type with common certainty of payoffs, where closeness is in the product topology in the universal belief space of Mertens and Zamir (1985).

⁹Angeletos and Werning (2006) give a price revelation foundation for the assumption that idiosyncratic shocks should have no less variation than common shocks.

¹⁰We allow for a qualitatively richer class of rank beliefs than the first generation literature. The existing

possibility that information alone does not determine behavior and that some other factor or factors determine equilibrium choice—our focus was on hysteresis as that factor.

This approach generated three novel and intuitive predictions. First, if we look at the relationship between fundamentals and outcomes, play must shift when once fundamentals cross a fundamental threshold that arises before an action becomes dominant. Second, large shocks can trigger a shift before that threshold is reached. And third, those shifts can only occur once an action is latent—i.e., the best response to uniform rank beliefs and thus the first generation global game prediction.¹¹

A Omitted Proofs

We start with some useful notation. For any two functions h_1 and h_2 from reals to reals, we define *convolution* $h_1 * h_2$ of h_1 and h_2 by

$$h_1 * h_2(z) = \int h_1(\varepsilon) h_2(z - \varepsilon) d\varepsilon. \quad (12)$$

Observe that

$$R(z) = \frac{Ff * g(z)}{f * g(z)}. \quad (13)$$

Since $F(-\varepsilon) = 1 - F(\varepsilon)$ and f and g are even functions, we have the following useful properties:

$$f * g(z) = f * g(-z); \quad (14)$$

$$R(-z) = \frac{(1 - F)f * g(z)}{f * g(z)}, \quad (15)$$

where $1 - F$ is the complementary cdf. The first property states that $f * g$ is even, and the second property states that $R(-z)$ is simply computed by using the complementary cdf. Hence,

$$\begin{aligned} R(z) - R(-z) &= \frac{(2F - 1)f * g(z)}{f * g(z)} = \frac{\int_{-\infty}^{\infty} (2F(\varepsilon) - 1) f(\varepsilon) g(z - \varepsilon) d\varepsilon}{f * g(z)} \\ &= \frac{\int_0^{\infty} (2F(\varepsilon) - 1) f(\varepsilon) (g(z - \varepsilon) - g(z + \varepsilon)) d\varepsilon}{f * g(z)} \end{aligned} \quad (16)$$

literature exclusively focusses on two cases (see Morris and Shin (2003) for a discussion of both cases). First, the case where we fix the distribution of common shocks / public signals and let the noise in idiosyncratic shocks / private signals go to zero; in this case there is common certainty of uniform rank beliefs in the limit. Second, the case where both common shocks / public signals and idiosyncratic shocks / private signals are normally distributed. As we noted in Section 7, rank beliefs are monotonic in this case and our large shock results cannot arise.

¹¹One informal explanation of equilibrium shifts is that a public shock coordinates players on the shift (Chwe (2013)). However, this explanation does not explain the direction of the shift nor identify the necessity of latency.

where the first equality is by (13), (14) and (15); the second equality is by definition of convolution, and the last property is by the fact that $2F - 1$ is an odd function while f is even.

Proof of Lemma 1. (Symmetry) By (15),

$$R(-z) = \frac{(1-F)f * g(z)}{f * g(z)} = \frac{f * g(z) - Ff * g(z)}{f * g(z)} = 1 - R(z).$$

(Single Crossing) For any $z > 0$, observe that $g(z - \varepsilon) - g(z + \varepsilon) \geq 0$ and the inequality is strict with positive probability; equality holds only if g is constant over the relevant range. Hence, by (16), $R(z) - R(-z) > 0$. Since $R(-z) = 1 - R(z)$, this also implies that $R(z) > 1/2 > R(-z)$.

(Uniform Limit Rank Beliefs) Fix any $\epsilon \in (0, 1)$. Since g has regularly varying tails (5), there exist $\beta > 0$ and η_0 such that for all $\eta' > \eta \geq \eta_0$,

$$\frac{g(\eta)}{g(\eta')} \leq (1 + \epsilon/2) (\eta/\eta')^{-\beta}. \quad (17)$$

Fix also $\gamma > 0$ such that

$$(1 + \epsilon/2) \left(\frac{1 - \gamma}{1 + \gamma} \right)^{-\beta} < 1 + \epsilon. \quad (18)$$

Now, by definition, for any $z > 0$,

$$R(z) \leq (I_1 + I_2) / I_3$$

where

$$\begin{aligned} I_1 &= \int_{-\gamma z}^{\gamma z} f(\varepsilon) F(\varepsilon) g(z - \varepsilon) d\varepsilon \leq \frac{1}{2} (F(\gamma z) - F(-\gamma z)) g(z - \gamma z), \\ I_2 &= \int_{\varepsilon \notin (-\gamma z, \gamma z)} f(\varepsilon) F(\varepsilon) g(z - \varepsilon) d\varepsilon \leq f(\gamma z), \\ I_3 &= \int_{-\gamma z}^{\gamma z} f(\varepsilon) g(z - \varepsilon) d\varepsilon \geq (F(\gamma z) - F(-\gamma z)) g(z + \gamma z). \end{aligned}$$

Combining the above inequalities, we conclude that

$$R(z) \leq \frac{1}{2} \frac{g(z - \gamma z)}{g(z + \gamma z)} + \frac{f(\gamma z)}{(F(\gamma z) - F(-\gamma z)) g(z + \gamma z)}. \quad (19)$$

Now, by (17) and (18),

$$\frac{g(z - \gamma z)}{g(z + \gamma z)} \leq \frac{1}{2} (1 + \epsilon/2) \left(\frac{1 - \gamma}{1 + \gamma} \right)^{-\beta} < 1/2 + \epsilon/2$$

for any $z > \eta_0 / (1 - \gamma)$. Moreover, by (6), there exists $\bar{z} > \eta_0 / (1 - \gamma)$ such that for all $z > \bar{z}$,

$$\frac{f(\gamma z)}{(F(\gamma z) - F(-\gamma z)) g(z + \gamma z)} < \epsilon/2.$$

Substituting, the two displayed inequalities in (19), we obtain $R(z) < 1/2 + \epsilon$ for all $z > \bar{z}$, as desired. ■

We next prove Propositions 1 and 2. Towards this goal, we present the following useful lemma.

Lemma 4 *We have*

$$\begin{aligned} x^*(\theta_{-1}) &> 1/2 > \theta_{-1} \quad \text{for all } \theta_{-1} < \underline{\theta} \equiv 1 - \bar{\theta}; \\ x^{**}(\theta_{-1}) &> \theta_{-1} > x^*(\theta_{-1}) \quad \text{for all } \theta_{-1} \in [\underline{\theta}, \bar{\theta}]; \\ x^{**}(\theta_{-1}) &< 1/2 < \theta_{-1} \quad \text{for all } \theta_{-1} > \bar{\theta}. \end{aligned}$$

Proof. By symmetry, we have

$$x^*(\theta_{-1}) = x^{**}(1 - \theta_{-1}).$$

Hence, it suffices to focus on $\theta_{-1} \geq 1/2$ and x^{**} . Take any $\theta_{-1} > \bar{\theta}$. By definition of $\bar{\theta}$, for any $z \geq 0$,

$$R(z) < \theta_{-1} + \sigma z,$$

showing that $z^{**}(\theta_{-1}) < 0$. Therefore, by Lemma 1,

$$x^{**}(\theta_{-1}) = R(z^{**}(\theta_{-1})) < 1/2.$$

Likewise, for $\theta_{-1} \leq \bar{\theta}$, there exists $z > 0$ such that $R(z) \geq \theta_{-1} + \sigma z$, showing that $z^{**}(\theta_{-1}) > 0$, and therefore

$$x^{**}(\theta_{-1}) = \theta_{-1} + \sigma z^{**}(\theta_{-1}) > \theta_{-1}.$$

■

Proof of Proposition 1. To prove the sufficiency, first observe that for any θ_{-1} ,

$$x^{**}(\theta_{-1}) = R(z^{**}(\theta_{-1})) \leq \bar{R},$$

where the equality is by definition of $x^{**}(\theta_{-1})$ and the inequality is by definition of \bar{R} . Now, invest is MUR whenever $\theta > \bar{R}$. Hence, assume that $\bar{R} \geq \theta > 1/2$ and (11) holds, so that $\eta = (\theta - \theta_{-1})/\sigma > \bar{z}(\theta)$ —as in the left panel of Figure 3. Then, for all $z \geq \eta$ we have

$$R(z) < \theta = \theta_{-1} + \sigma\eta \leq \theta_{-1} + \sigma z,$$

where the strict inequality is by definition of $\bar{z}(\theta)$ and $z > \bar{z}(\theta)$. Hence, $z^{**}(\theta_{-1}) < \eta$, showing that

$$x^{**}(\theta_{-1}) = \theta_{-1} + \sigma z^{**}(\theta_{-1}) < \theta_{-1} + \sigma\eta = \theta. \quad (20)$$

Therefore, invest is MUR at θ under θ_{-1} .

To prove the converse, take any $\theta_{-1} \leq \bar{R} - \sigma\bar{z}(\bar{R})$. Since $\theta_{-1} \leq \bar{\theta}$, by Lemma 4, $x^{**}(\theta_{-1}) > \max\{\theta_{-1}, 1/2\}$. Hence, if invest is not latent (i.e. $\theta \leq 1/2$), then $x^{**}(\theta_{-1}) > \theta$, and therefore

invest is not MUR. Now, assume that invest is latent (i.e. $\theta > 1/2$) but inequality (11) does not hold—as in the right panel of Figure 3. Then, since $\theta - \theta_{-1} \leq \sigma \bar{z}(\theta)$,

$$\eta \leq \bar{z}(\theta). \quad (21)$$

We **claim** that, if in addition R is single peaked, then (21) implies that $x^{**}(\theta_{-1}) \geq \theta$, and therefore invest is not MUR. To prove the **claim** that $x^{**}(\theta_{-1}) \geq \theta$, suppose $x^{**}(\theta_{-1}) < \theta$ and equivalently

$$z^{**}(\theta_{-1}) < \eta. \quad (22)$$

Now, since $\theta_{-1} \leq \bar{R} - \sigma \bar{z}(\bar{R})$, it must be that $z^{**}(\theta_{-1}) \geq \bar{z}(\bar{R})$, and thus, by (21) and (22), we have

$$\bar{z}(\bar{R}) \leq z^{**}(\theta_{-1}) < \eta \leq \bar{z}(\theta).$$

However, since R is single-peaked with a peak at $\bar{z}(\bar{R})$, this implies that

$$x^{**}(\theta_{-1}) = R(z^{**}(\theta_{-1})) > R(\bar{z}(\theta)) = \theta,$$

contradicting that $x^{**}(\theta_{-1}) < \theta$. ■

Proof of Proposition 2. Set

$$\Delta = \min \{x^{**}(\bar{\theta}) - \bar{\theta}, \bar{\theta} - 1/2\}. \quad (23)$$

Observe that

$$\min_{\theta \geq \underline{\theta}} (\theta - x^*(\theta)) = \underline{\theta} - x^*(\underline{\theta}) = x^{**}(\bar{\theta}) - \bar{\theta} = \min_{\theta \leq \bar{\theta}} (x^{**}(\theta) - \theta) > 0, \quad (24)$$

where the first and the last equalities are by the fact that $\theta - x^*(\theta)$ is increasing while $x^{**}(\theta) - \theta$ is decreasing, and the middle equality is by symmetry. By 4, $x^{**}(\bar{\theta}) - \bar{\theta} > 0$.

Consider any $\theta_{-1} > \bar{\theta}$. Clearly, by definition (23),

$$\theta_{-1} > \bar{\theta} \geq \Delta + 1/2.$$

Hence, for any θ with $|\theta - \theta_{-1}| \leq \Delta$, we have

$$\theta \geq \theta_{-1} - \Delta > 1/2 > x^{**}(\theta_{-1}),$$

showing that invest is MUR at θ under θ_{-1} . (Here, the last inequality is by Lemma 4.)

Now consider any $\theta_{-1} \leq \bar{\theta}$. By (23) and (24),

$$x^{**}(\theta_{-1}) - \theta_{-1} \geq x^{**}(\bar{\theta}) - \bar{\theta} \geq \Delta.$$

Hence, for any θ with $|\theta - \theta_{-1}| \leq \Delta$, we have

$$\theta \leq \theta_{-1} + \Delta \leq x^{**}(\theta_{-1}),$$

showing that invest is not MUR at θ under θ_{-1} .

The second statement in the proposition follows from the first one by symmetry. ■

Proof of Proposition 4. Take Δ as in the proof of Proposition 2. By Proposition 2, if $|\theta_t - \theta_{t-1}| \leq \Delta$ and $\theta_{t-1} > \bar{\theta}$, then invest is MUR at θ_t under θ_{t-1} , showing by Lemma 2 that a majority invests under $s(h_t, \cdot)$. Therefore, there is majority investment at (h_t, θ_t) under s . By symmetry, this also implies that there is minority investment at any (h_t, θ_t) with $|\theta_t - \theta_{t-1}| \leq \Delta$ and $\theta_{t-1} < \underline{\theta}$. Finally, if $\theta_{t-1} \geq \underline{\theta}$ and $|\theta_t - \theta_{t-1}| < \Delta$, by (24), we have

$$\theta_t - x^*(\theta_{t-1}) > \theta_{t-1} - x^*(\theta_{t-1}) - \Delta \geq 0.$$

Hence,

$$\alpha(h_t, \theta_t | s^*) = 1 - F\left(\frac{x^*(\theta_{t-1}) - \theta_t}{\sigma}\right) > 1/2,$$

showing that there is majority investment at (h_t, θ_t) . ■

Proof of Lemma 3. Consider any θ_{-1} and θ'_{-1} with $\theta_{-1} > \theta'_{-1}$. Observe that

$$R(z^{**}(\theta_{-1})) = \theta_{-1} + \sigma z^{**}(\theta_{-1}) > \theta'_{-1} + \sigma z^{**}(\theta_{-1}).$$

Moreover, since R is bounded and $\theta'_{-1} + \sigma z$ goes to infinity, there exists $\tilde{z} > z^{**}(\theta_{-1})$ such that

$$R(\tilde{z}) < \theta'_{-1} + \sigma \tilde{z}.$$

Since R is continuous, by the intermediate value theorem, this implies that there exists $z > z^{**}(\theta_{-1})$ with $R(z) = \theta'_{-1} + \sigma z$, showing that $z^{**}(\theta'_{-1}) > z^{**}(\theta_{-1})$. One can similarly prove that $z^*(\theta'_{-1}) > z^*(\theta_{-1})$. ■

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