# Firm Volatility in Granular Networks[*]

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## Abstract

We propose a model of firm volatility based on customer-supplier connectedness. In our model, customer growth rate shocks influence the growth rates of their suppliers, larger suppliers have more customers, and the strength of a customer-supplier link depends on the size of the customer firm. When the size distribution becomes more dispersed, economic activity is concentrated among a smaller number of large customers, the typical supplier becomes less diversified and its volatility increases. We document new stylized facts that are consistent with the model. At the macro level, we show that the firm size distribution and firm volatility distribution evolve together. Economy-wide firm size concentration explains common movements in firm-level volatility. At the micro level, we show that the sales network structure is an important determinant of firm-level volatility.

*JEL: G12, G15, F31.*

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# 1  Introduction

In this paper we document a range of new empirical facts regarding firm size and firm volatility. At the macroeconomic level, the economy-wide firm size distribution and the cross section distribution of firm-level growth rate volatilities evolve in unison. At the microeconomic level, volatilities across firms covary strongly, and a firm's volatility is closely associated with the structure of its economic linkages with other firms.

To explain these facts, we develop a theoretical framework in which shocks propagate through the economy via firm-level customer-supplier networks. The model can explain the aggregate empirical relationships between the firm size distribution and the volatility distribution. Our model ascribes the secular increase in U.S. firm-level volatility documented by Campbell, Lettau, Malkiel, and Xu (2001) to economy-wide increases in the concentration of customer networks. The model also replicates the factor structure in firm-level volatility.

The volatility of individual stock returns varies greatly over time (Lee and Engle (1993)) and across different firms (Black (1976), Christie (1982), and Davis, Haltiwanger, Jarmin, and Miranda (2007)). While much progress has been made in describing the time-series dynamics of volatility, our understanding of the underlying determinants of stock return volatility is limited.

We usually think of exogenous variation in risk and uncertainty at the firm level as the driving forces of behind variation in aggregate volatility.[1] In this paper, we approach firms themselves as aggregators of downstream shocks to other firms in their customer network. We show how both the size distribution and network structure interact to produce cross section heterogeneity and time series dynamics in firms' growth rate volatility. By studying firms as shock aggregators, our paper brings volatility back inside the model.

Our network model of firm growth makes three critical assumptions. First, customer

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[1]For example, Gabaix (2011) shows how the size distribution influences aggregate fluctuations holding firms' volatilities fixed. Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) study how aggregate shocks are influenced by network structures. These authors model the economy as an aggregator of firm-level or sector-level shocks.

growth rate shocks influence the growth rates of their suppliers. When a customer suffers an adverse productivity shock, it purchases less from its suppliers, and thus a portion of its shock is transmitted upstream. Second, larger suppliers have a higher probability of being connected with customers. Hence, large firms typically supply to a higher number of customers. Third, the weight of a customer-supplier link depends on the size of the customer firm. A large customer will have a stronger connection with its suppliers than do smaller customers because, for example, it accounts for a large fraction of the supplier's total sales.[2]

These simple assumptions have a rich set of macroeconomic and microeconomic implications for growth rate volatility. First, larger firms have lower volatility because they are connected to more customers. Firms are aggregators of downstream shocks to the other firms in their customer network, thus a broader customer base improves diversification.

Second, firms with more concentrated customer networks have higher volatility. In the model, sizes of customers determine the strength of connections. Therefore, a supplier's network will be more concentrated if there is high dispersion in its customers' sizes. When a supplier has some customers that are much larger than others, the large customers' shocks will exert an outsized influence on the supplier. A supplier with a very concentrated customer base will be poorly diversified and have high volatility even if it has a large number of customers.

Gabaix (2011) argues that there is a "granularity" effect at the economy-level – a small number of large firms drive aggregate volatility since aggregation implies that shocks are size-weighted. Our model embeds the firm-level analogue of granularity – a small number of large customers drive most of the variation of individual firms. As a given supplier's customer size distribution becomes more concentrated, the standard $1/\sqrt{N}$ diversification of shocks becomes impaired, and the supplier's volatility increases as a result.

This framework predicts a close association between the cross section distributions of size

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[2]In the language of networks, our customer-supplier model is a directed random graph in which a node's expected out-degree depends on the size of the node, and in which connections are weighted by the size of the receiver node.

and volatility. The size distribution at any point in time determines the network structure. Given firm sizes, the probability distribution of connections for all nodes in the network is pinned down, as are the weights of directed links between nodes (conditional on a link existing). The network structure, in turn, determines the volatility distribution. Our model's random graph structure means that each supplier's network is a random draw from the entire size distribution and therefore inherits the concentration of the entire size distribution (in expectation). Because all firms draw from the same economy-wide size distribution, their customer networks have equal concentration, in expectation. Cross section differences in volatility are therefore mostly driven by their differences in size rather than differences in customer concentration.

Variation in the size distribution, combined with the determination of volatilities within the network mechanism, induces endogenous joint variation in the size and volatility distribution. When the size distribution widens, total economic activity is concentrated among a smaller number of firms. This affects the diversifiability of shocks for all firms, but more so for small firms who have few shocks to diversify in the first place. As a result, an approximate factor structure in total and idiosyncratic firm-level volatility arises in which the factor is the concentration of the entire size distribution.

Consistent with these predictions, Kelly, Lustig and Van Nieuwerburgh (2012) uncover a factor structure in the volatility of stock returns from 1926-2010. Even after removing all of the common variation in returns, the volatility of the residuals inherits the same factor structure as total volatility. The volatility factor accounts for between 30% and 40% of the variation in firm-level volatility. They also show that the same factor structure appears in the volatility of sales and earnings growth.[3]

We document a range of new stylized facts, both macroeconomic and microeconomic, that

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[3]Similarly, Engle and Figlewski (2012) document a common factor in option-implied volatilities since 1996, and Barigozzi, Brownlees, Gallo, and Veredas (2010) and Veredas and Luciani (2012) examine the factor structure in realized volatilities of intra-daily returns since 2001. We provide an economic explanation for a common factor in firm-level volatility, and argue that the cross-sectional dispersion of firm size is natural candidate for this volatility factor.

are consistent with the economic network model. Our macroeconomic facts link the cross section firm size distribution with the cross section of firm volatility. When the firm size distribution is highly dispersed, the mean and dispersion in firms' volatilities are also high.[4] An important validation of our model is that firm size dispersion predicts average volatility and volatility dispersion, but not the reverse. Firm volatility (both total and idiosyncratic) has a strong factor structure, and concentration in the firm size distribution is a successful factor for describing common variation in volatilities across firms. We show this both for firms' quarterly sales growth volatility and also for firms' return volatility (which is measured far more accurately).

We find that time series dynamics in the firm size distribution can account for trends in average firm-level volatility that have been studied by Campbell et al. (2001). The volatility of firm-level stock returns has increased from an average of 26% per year during the 1950's to 63% per year since 1990 (see, for example, Campbell, Lettau, Malkiel, and Xu (2001)).[5]

At the microeconomic level, we begin by documenting the importance of firm sizes (defined as a firm's total sales) for describing firm-level linkages in US customer-supplier networks. The customer-supplier data has two clear empirical features. First, larger suppliers have a higher customer count. Second, for a given supplier, larger customers account for a larger fraction of the supplier's sales than do smaller customers. These findings help validate our network model assumptions.

We then calibrate our network model to evaluate its ability to match a range of moments in the customer-supplier linkage data. These moments include correlations between a firm's size, its growth rate volatility, its number of customers, and the concentration of its customer base, as well as cross section quantiles of the size, volatility and out-degree distributions. The model does well provided that we have strong network feedback effects, but to match

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[4]Size and volatility distributions are approximately log normal, thus we track their joint dynamics by comparing their first two moments over time.

[5]Since firm-level volatility of asset returns measures the uncertainty faced by the managers of firms (Leahy and Whited (1996)), this represents a large increase in uncertainty. Uncertainty has a major impact on employment and investment (Bloom, Bond, and Van Reenen (2007), Bloom (2009), Stokey (2012) ).

the levels of volatility in the data, the model requires too much concentration in customers network.

We analyze the determinants of firm-level volatility in light of the network model. We find that firm size and, more tellingly, the concentration of firms' customer networks, explains a substantial fraction of the variation in volatility across firms and over time. Each of these possess stronger explanatory power than the range of other volatility determinants that have been studied in the literature, including industry concentration and competition, R&D intensity, equity ownership composition, and firm age or cohort effects.

The rest of the paper is structured as follows. Section 2 describes the generic network model and explores its volatility implications. Section 3 imposes more structure on the nature of linkages, and it describes the factor structure in volatility that is generated by our model. Section 4 presents macroeconomic evidence of the association between the firm size distribution and the firm volatility distribution. Section 5 describes micro evidence for the association between firms' customer network structures and their volatility. Section 6 calibrates the model that to match key features of the macro evidence linking size and volatility distributions, and the micro network evidence. All of the proofs are in the appendix.

# 2    A Network Model of Firm Growth

We develop a dynamic network model of customer-supplier relations. In this directed network, firms are exposed to their own idiosyncratic growth rate shocks, and also exposed to growth rates of their customers. Shocks are propagated upstream through the network and thus idiosyncratic shocks effectively become systematic.

## 2.1    Firm Growth

Define $S_{i,t}$ as the size of firm $i$, and its growth rate as $g_{i,t+1}$, where $S_{i,t+1} = S_{i,t} \exp(g_{i,t+1})$. Firm $i$'s growth rate is defined as a linear combination of a firm-specific shock and a weighted

average of the growth rates of firms $i$'s customers:

$$g_{i,t+1} = \mu_g + \gamma \sum_{j=1}^{N} w_{i,j,t}\, g_{j,t+1} + \varepsilon_{i,t+1}. \tag{1}$$

The parameter $\gamma$, which is assumed to be less than one in magnitude, governs the rate of decay as a shock propagates through the network. The weight $w_{i,j,t}$ governs how strongly firm $i$'s growth rate is influenced by the growth rate of firm $j$, and $\gamma \sum_{j=1}^{N} w_{i,j,t}\, g_{j,t+1}$ is the network's aggregate impact on firm $i$. If $i$ and $j$ are not connected then $w_{i,j,t} = 0$. Any firm $j$ that possesses a non-zero weight $w_{i,j,t}$ is interpreted as a customer of firm $i$. The full matrix of connection weights is $\boldsymbol{W}_t = [w_{i,j,t}]$. The structure in (17) implies that all rows of $\boldsymbol{W}_t$ sum to one. Its largest eigenvalue $\lambda_{max}$ is one.

Each firm $i$ experiences an idiosyncratic, homoskedastic growth rate shock $\varepsilon_{i,t+1} \sim N(0, \sigma_\varepsilon)$, and by convention the weight on the firm's own growth is always zero ($w_{i,i,t} = 0\, \forall\, t, i$). All of the dynamics in the volatility of growth rates will arise endogenously.

This is a directed network. Equation (1) says that shocks are propagated upstream, from customer to supplier, rather than downstream. Shea (2002) lists a class of equilibrium models in which final demand shocks are transferred upstream. Our model fits into this class.[6] Our specification emphasizes demand shocks over supply shocks as drivers of the network component of firm-level volatility. Our specification abstracts from a major source of leakage in the network: retailers. However, if markets are incomplete, retailers are probably exposed to some of the supplier's shocks indirectly because of the labor income and wealth shocks experienced by consumers (see section B in the separate appendix).

Connections are not generally symmetric, thus firm $j$ can be a customer of $i$ without $i$ being a customer of $j$.

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[6]Acemoglu et al. (2012) derive a static version of (2) as the equilibrium outcome in a multi-sector production economy in a constant-returns to scale economy who has Cobb-Douglas preferences defined over all of the $N$ different commodities produced in this economy. Theirs is also a directed network, but in their version the productivity shocks are transferred downstream from suppliers to customers.

## 2.2 Volatility of Firm Growth

In matrix notation, $\boldsymbol{g}_{t+1}$ is the $N \times 1$ vector of growth rates and $\boldsymbol{\varepsilon}_{t+1}$ the vector of shocks, and (1) is

$$\boldsymbol{g}_{t+1} = \boldsymbol{\mu}_g + \gamma \boldsymbol{W}_t \boldsymbol{g}_{t+1} + \boldsymbol{\varepsilon}_{t+1}, \quad \text{where } \boldsymbol{\varepsilon}_{t+1} \sim N\left(0, \sigma_\varepsilon^2 \boldsymbol{I}\right). \tag{2}$$

As long as $\gamma$ is smaller than one in magnitude,[7] the matrix $(\boldsymbol{I} - \gamma \boldsymbol{W}_t)$ is nonsingular and $\boldsymbol{g}_{t+1}$ can be restated as

$$\boldsymbol{g}_{t+1} = (\boldsymbol{I} - \gamma \boldsymbol{W}_t)^{-1}\left(\boldsymbol{\mu}_g + \boldsymbol{\varepsilon}_{t+1}\right). \tag{3}$$

This variance-covariance matrix of $\boldsymbol{g}_{t+1}$ is therefore

$$\boldsymbol{V}_t\left(\boldsymbol{g}_{t+1}\right) = \sigma_\varepsilon^2 \left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t'\right)^{-1}. \tag{4}$$

These growth dynamics generate dynamics to the relative size (or shares) of firms that are similar to those explored by Menzly, Santos, and Veronesi (2004) and Santos and Veronesi (2006) in a series of papers. However, there is no mean reversion built into our shares. Furthermore, only the customer shares in $\boldsymbol{W}_t$ are relevant for the cash flow dynamics of a firm, not the firm size relative to the entire economy.

Let $\tilde{\boldsymbol{S}}$ denote the vector of relative sizes. Then we can define the aggregate growth rate of this economy as

$$g_{a,t+1} = \tilde{\boldsymbol{S}}_t' \boldsymbol{g}_{t+1} = \tilde{\boldsymbol{S}}_t' \left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{\mu}_g + \boldsymbol{\varepsilon}_{t+1}\right). \tag{5}$$

The variance of the aggregate growth rate is given by

$$V_t(g_{a,t+1}) = \sigma_\varepsilon^2 \tilde{\boldsymbol{S}}_t' \left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t'\right)^{-1} \tilde{\boldsymbol{S}}_t. \tag{6}$$

A standard approach to evaluating systematic versus idiosyncratic risk is to run factor

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[7]In general, $|\gamma|$ must be smaller than the largest eigenvalue of $\boldsymbol{W}_t$. Because we have normalized the row sums of $\boldsymbol{W}_t$ to one, one is its largest eigenvalue.

model regressions. The primary factor is typically a size-weighted aggregate of the left hand side variables. It is useful to examine how the network model impacts this type of analysis. Let $\boldsymbol{\beta}$ denote the $N \times 1$ vector of betas estimated over some sample. Then the residual growth rate is defined as:

$$\boldsymbol{g}_{t+1}^e = \boldsymbol{g}_{t+1} - \boldsymbol{\beta} g_{a,t+1} = \left(\boldsymbol{I} - \boldsymbol{\beta} \tilde{\boldsymbol{S}}_t'\right)\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{\mu}_g + \boldsymbol{\varepsilon}_{t+1}\right) \tag{7}$$

The variance of the residual growth rate is given by:

$$V_t(g_{t+1}^e) = \sigma_\varepsilon^2 \left(\boldsymbol{I} - \boldsymbol{\beta} \tilde{\boldsymbol{S}}_t'\right)\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t'\right)^{-1}\left(\boldsymbol{I} - \tilde{\boldsymbol{S}}_t \boldsymbol{\beta}'\right). \tag{8}$$

The variation in $\boldsymbol{W}_t$ and $S_t$ will induce heteroskedasticity in residuals, even though the underlying innovations are i.i.d.

To better understand the relation between the volatility and the network structure, we start by focussing on first-order network effects.

## 2.3 First-order Effects on Volatility

The first order approximation to the full network model allows us to develop intuition via analytical expressions for firm volatility in the model. Provided that $|\gamma| < 1$, the first order approximation of the inverse term in $\boldsymbol{g}_{t+1} = \left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1}\left(\boldsymbol{\mu}_g + \boldsymbol{\varepsilon}_{t+1}\right)$ is given by

$$
\begin{aligned}
\left(\boldsymbol{I} - \gamma \boldsymbol{W}_t\right)^{-1} &= \boldsymbol{I} + \gamma \boldsymbol{W}_t + \gamma^2 \boldsymbol{W}_t^2 + \gamma^3 \boldsymbol{W}_t^3 + \cdots \\
&\approx \boldsymbol{I} + \gamma \boldsymbol{W}_t.
\end{aligned}
$$

Hence, we have the following first-order approximations of the growth rate and the variance of the growth rate

$$\boldsymbol{g}_{t+1} \approx \left(\boldsymbol{I} + \gamma \boldsymbol{W}_t\right)\left(\boldsymbol{\mu}_g + \boldsymbol{\varepsilon}_{t+1}\right). \tag{9}$$

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In this first order approximation, firm $i$'s growth becomes

$$g_{i,t+1} \approx (1+\gamma)\,\mu_g + \varepsilon_{i,t+1} + \gamma \sum_j w_{i,j,t}\varepsilon_{j,t+1}. \tag{10}$$

The variance for this first order approximation is given by:

$$V_t\left(g_{i,t+1}\right) \approx \sigma_\varepsilon^2\left(1+\gamma^2 H_{i,t}\right), \tag{11}$$

where $H_{i,t} \equiv \sum_{j=1}^{N} w_{i,j,t}^2$ is the size Herfindahl of firms in supplier $i$'s customer network. Hence, to a first order, the variance of the firm's growth rate is determined by its customer network Herfindahl $H_{i,t}$, the volatility of the underlying innovations $\sigma_\varepsilon^2$, and the rate of shock decay in the network $\gamma$.

All dynamics in volatility, according to this model, are driven by the respective supplier's customer concentration component, $H_{i,t}$. This will allow us to identify the source of common volatility dynamics among *all* firms. It is easy to show that the first-order approximation of volatility delivers a lower bound for the true variance in our full-fledged network model.

**Proposition 1.** *The average variance is bounded below by the average customer Herfindahl:*

$$\frac{1}{N}\sum_i V_t\left(g_{i,t+1}\right) \geq \sigma_\varepsilon^2\left(1+\gamma^2\frac{1}{N}\sum_i H_{i,t}\right) \tag{12}$$

This follows because the higher-order feedback effects that we abstract from ion the first-order approximation can only act to increase firm-level volatility.

## 2.4  Higher-order Network Effects on Volatility

The strength of higher-order network effects depends on $\gamma$, which governs the rate of shock decay, and the properties of the weighted adjacency matrix $\boldsymbol{W}_t$. We can put bounds on these

effects in symmetric versions of our model. To develop intuition, it is helpful to consider two polar cases. The first case that we consider features maximally strong feedback effects, but no diversification benefits from linkages. In such a network, each firm $i$ only has one connection. The corresponding $\boldsymbol{W}_t$ matrix has ones above the main diagonal and zeros elsewhere, except in the $(N, 1)$ position.

**Proposition 2.** *In a symmetric network with minimum diversification and maximum concentration, the variance of a typical firm's growth rate is:*

$$V\left(g\right) = \frac{\sigma_\epsilon^2 (1 - \gamma^{2N})}{(1 - \gamma^2)(1 - \gamma^N)^2} \tag{13}$$

As $N \to \infty$, we get an expression for the variance of the firm that is identical to that of an AR(1) time-series process:

$$V\left(g\right) = \frac{\sigma_\epsilon^2}{1 - \gamma^2}. \tag{14}$$

As $\gamma$ approaches 1, the volatility in the network explodes because shocks do not decay as they propagate through the network. Fixing $\gamma$, expression (13) represents an upper bound on the volatility in symmetric networks in our model.

On the other hand of the spectrum, we consider a network in which each firm is maximally diversified. Each firm is connected equally to all other firms in the network. In this case, there is no concentration in suppliers' customer bases. The corresponding $\boldsymbol{W}_t$ matrix has zeros on the main diagonal and $\frac{1}{N-1}$ elsewhere.

**Proposition 3.** *In a symmetric network with maximum diversification and minimum concentration, the variance of a typical firm's growth rate is*

$$V(g) = \sigma_\varepsilon^2 \left[ \frac{\left( \frac{N-2}{N-1}\gamma - 1 \right)^2 + \frac{1}{N-1}\gamma^2}{\left( \frac{1}{N-1}\gamma^2 + \frac{N-2}{N-1}\gamma - 1 \right)^2} \right] \tag{15}$$

As $N \to \infty$, the variance of all firms in the diversified network converges to the original shock variance $\sigma_\epsilon^2$.

These two cases, maximal concentration and maximal diversification, define a lower and upper bound on the variance of firms in a symmetric network For all $i$ and $t$, we have

$$\sigma_\varepsilon^2 \left[ \frac{\left(\frac{N-2}{N-1}\gamma - 1\right)^2 + \frac{1}{N-1}\gamma^2}{\left(\frac{1}{N-1}\gamma^2 + \frac{N-2}{N-1}\gamma - 1\right)^2} \right] \leq V(g_{i,t}) \leq \frac{\sigma_\epsilon^2(1 - \gamma^{2N})}{(1 - \gamma^2)(1 - \gamma^N)^2}. \tag{16}$$

As $N \to \infty$, this interval converges to $[\sigma_\epsilon^2, \sigma_\epsilon^2/(1 - \gamma^2)]$. As the size distribution and, hence, the network structure evolve, there will be variation in firms' growth rate variance within these bounds. If the network is sufficiently concentrated and $\gamma$ approaches one, this range is arbitrarily large.

We cannot solve for the variance of the growth rates for general $\boldsymbol{W}$ matrices. However, we can derive tighter bounds than the first-order bounds by using an eigenvector decomposition. $\boldsymbol{W}$ can be interpreted as a transition probability matrix and the eigenvector associated with the unit eigenvalue (rescaled) represents its invariant distribution. We use $\widetilde{w}$ to denote this vector and we will refer to these as the ergodic business links. The elements of $\widetilde{w}$ can be interpreted as summarizing the total strength of all the direct and indirect business connections with $i$. Similarly, we use $\widetilde{w}^{-1}$ to denote the first column of the inverse of the matrix of eigenvectors.

**Proposition 4.** *There is a lower bound on the variance of firm $i$ given by:*

$$V(g_{i,t}) \geq \frac{\widetilde{w}_{t,i}^2 \sigma_\epsilon^2}{(1 - \gamma)^2} \sum_{k=1}^{N} (\widetilde{w}_{t,k}^{-1})^2$$

The interpretation is straightforward. The larger the ergodic strength of a firm's business connections, as summarized by $\widetilde{w}_{t,i}$, the higher its volatility.[8]

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[8]This bound is not a first-order approximation, but it incorporates all higher-order effects. This bound is

**Corollary 1.** *There is a lower bound on the the average firm-level variance determined by the Herfindahl of ergodic links, represented by the eigenvector associated with the unit eigenvalue:*

$$\frac{1}{N} \sum_i V_t(g_{i,t+1}) \geq \frac{\widetilde{H}_t}{(1-\gamma)^2} \sigma_\epsilon^2 \sum_{k=1}^{N} (\widetilde{w}_{t,k}^{-1})^2$$

*where $\widetilde{H}_t = \frac{1}{N} \sum_{i=1}^{N} \widetilde{w}_{t,i}^2$.*

This bound looks similar to the one derived in proposition 1. The difference is that this bound incorporates higher-order effects, as evidenced by the $1/(1-\gamma)$ term. The relevant Herfindahl is the one associated with the ergodic business linkages in the network.

# 3   Size and Volatility

Sofar we have not made any assumptions about the nature of these linkages. To match the cross-sectional evidence of customer-supplier relations, we introduce two size effects in our network model. The first concerns the strength of the linkages: larger customers account for a larger share of the supplier's sales and hence have a larger impact on the supplier's growth rate. The second size effect concerns the number of links: larger suppliers are connected to more firms. This model will match the relation between size, volatility, and customer network concentration in the data. In particular, it delivers a factor structure in volatility that resembles the one in the data.

## 3.1   Size Effects in Network Structure

The strength of $i$ and $j$'s customer-supplier relations depend on the size of customer $j$, and is defined as

$$w_{i,j,t} = \frac{b_{i,j,t} S_{j,t}}{\sum_{k=1}^{N} b_{i,k,t} S_{k,t}} \quad \forall i,j,t. \tag{17}$$

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obtained by only collecting the terms associated with the unit eigenvalue. When $\gamma$ is close to 1, this bound is likely to be tighter than the first-order bounds.

The existence of a link between $i$ and $j$ is captured by

$$b_{i,j,t} = \begin{cases} 1 & \text{if } i \text{ connected to } j \text{ at time } t \\ 0 & \text{otherwise,} \end{cases}$$

and, conditional on a link existing $(b_{i,j,t} = 1)$, the strength of that link is simply a size weight of customer $j$ relative to the total size of all of $i$'s customers, $S_{j,t}/\sum_{k=1}^{N} b_{i,k,t}S_{k,t}$. This weighting scheme assumes that larger customers have a larger impact on the supplier's growth rate. This assumption is the first size effect in our network, and we find strong evidence for it in Compustat data.

The linkage structure at time t is determined by the firm size distribution. Each element of the connections matrix, $\boldsymbol{B}_t = [b_{i,j,t}]$, is drawn from a Bernoulli distribution with $P(b_{i,j,t} = 1) = p_{i,t}$ and

$$p_{i,t} = f_i(\{S_{i,t}\}_{i=1}^{N}). \tag{18}$$

We consider a range of specifications for $p_{i,t}$ in our calibration, the main specification being $p_{i,t} = S_{i,t}/\left(Z\sum_j(S_{j,t})\right)$. According to this rule, the probability that supplier $i$ and customer $j$ are connected depends on the size of the supplier, not the customer. Furthermore, larger suppliers have more connections on average. This is the model's second size effect.

## 3.2 Factor Structure in Volatility

Kelly, Lustig and Van Nieuwerburgh (2012) document that there is a factor structure in firm-level volatility. Even after removing all of the common variation in returns, the volatility of the residuals is highly correlated across all firms. The average firm-level volatility accounts for between 30% and 40% in the overall variation in firm-level volatility, even after removing common sources of variation from returns. Common volatility fluctuations are shared by firms across characteristic groupings such as industry and size. We argue that network concentration is a natural candidate volatility factor.

### 3.2.1 Firm-level Volatility

The structure of our model embeds a similarity in customer network concentration for all suppliers, regardless of their size. Thus, the only difference in the network structure across suppliers of different size is the *number* of customers they have. This can be seen in the following large $N$ approximation.

**Proposition 5.** *In the first-order version of this network economy, firm-level volatility inherits an approximate factor structure, for large $N$:*

$$V_t\left(g_{i,t+1}\right) \approx \sigma_\varepsilon^2 \left(1 + \frac{1}{p_{i,t}}\gamma^2 H_t\right). \tag{19}$$

*where $H_t$ denotes the economy-wide Herfindahl.*

Because all firms, large and small, draw from the same economy-wide size distribution, their customer networks have approximately equal concentration. In (19), this is captured by all firms' volatilities being exposed to a common factor: the Herfindahl of the *entire* size distribution. Cross section differences in volatility are driven by their differences in size, rather than differences in customer concentration, and are captured by one over the linkage probability, $p_{i,t}$. Bigger firms have a higher probability of establishing connections, therefore they typically connect to more firms and achieve better diversification because of this. Therefore the factor loading $1/p_{i,t}$ is smaller for large firms, which scales down the level of their volatility relative to small firms, and also makes them less sensitive to fluctuations in the firm size distribution (they have less volatility of volatility).

**Corollary 2.** *If the firm size distribution is log-normal, then firm volatility in the first-order version of our economy for large networks is approximately given by:*

$$V(g_{i,t+1}) \approx \sigma_\varepsilon^2 \left(1 + \gamma^2 \frac{\exp\left(\sigma_{s,t}^2\right)}{N p_{i,t}}\right), \tag{20}$$

*where $\sigma_{s,t}^2$ denotes the cross-sectional variance of the size distribution.*

According to a first-order approximation of our network model, $\sigma^2_{s,t}$ is the natural candidate for the volatility factor provided that the size distribution is log normal.

**Proposition 6.** *In a first-order version of our network economy, if the size distribution is log-normal, the volatility distribution is also log-normal with mean $E\left[\log\left(V\left(g_{i,t+1}\right) - \sigma^2_\varepsilon\right)\right] \approx \log(\sigma^2_\varepsilon \gamma^2 Z) + \frac{3}{2}\sigma^2_{S,t}$ and volatility $V\left(\log\left(V\left(g_{i,t+1}\right) - \sigma^2_\varepsilon\right)\right) \approx \sigma^2_{S,t}$.*

These two equations imply that the dispersion in the size distribution forecasts the mean and variance of the volatility distribution. We show in data and model simulations that the distributions of firm size and firm volatility at any point in time are approximately log normal, and that variance in firm sizes Granger causes mean and variance of the the volatility distribution.

When the size distribution is not lognormal, other higher order moments may enter as factors well.

**Corollary 3.** *In general, the firm volatility in the first-order version of our economy for large networks is approximately given by:*

$$V(g_{i,t+1}) \approx \sigma^2_\varepsilon \left(1 + \frac{\gamma^2}{Np_{i,t}} \exp\left(\sum_{j=2}^{\infty} \frac{2^j - 2}{j!}\kappa_{s,j,t}\right)\right). \tag{21}$$

*where $\mu_{s,j,t} \equiv E[(\log S_t)^j]$ denotes the $j$-th central conditional moment and $\kappa_{1,s,t} = \mu_{1,s,t}$, $\kappa_{2,s,t} = \mu_{2,s,t}$, $\kappa_{3,s,t} = \mu_{3,s,t}$ and $\kappa_{4,s,t} = \mu_{4,s,t} - 3\mu^2_{2,s,t}$. In the case of log-normality, the sum collapses to $\sigma^2_{s,t}$.*

As expected, positive skewness and positive excess kurtosis in the distribution of log size will tend to increase firm-level volatility. These higher order moments could enter as additional volatility factors.

### 3.2.2 Residual Volatility

In the literature, idiosyncratic volatility is constructed by first removing the correlated component of growth rates or returns with a statistical factor model, then calculating the volatil-

ities of the residuals (see, e.g., Bekaert, Hodrick, and Zhang (2010)). In a sparse random network model, a standard factor regression is mis-specified. There is no dimension-reducing factor that can capture commonalities in growth rates since, by virtue of the network, *every* firm's shock may be systematic. A factor model will be mis-specified by definition, and a telltale sign of the mis-specification will be residuals inheriting a volatility factor structure that looks very similar to the factor structure for total volatility.

**Proposition 7.** *In a first-order version of our network economy, the volatility of residuals also inherits an approximate factor structure:*

$$V_t(g_{i,t+1}^{res}) \approx \sigma_\varepsilon^2 \left(1 + \frac{1}{p_{i,t}} \gamma^2 H_t - \left[\frac{S_{i,t}}{NE[S_{i,t}]} + \gamma H_t\right]^2\right). \tag{22}$$

### 3.2.3 Aggregate Volatility

Finally, we consider the impact on aggregate volatility. In the first-order version of our economy, aggregate volatility also is governed by the same factor structure.

**Proposition 8.** *In the first-order version of this network economy, aggregate volatility inherits an approximate factor structure:*

$$V(g_{a,t+1}) \approx \sigma_\varepsilon^2 H_t (1 + \gamma)^2. \tag{23}$$

Equation (23) predicts a strong link between dispersion in firm size (summarized by the Herfindahl of the firm size distribution, $H_t$) and volatility of aggregate growth in the first-order version of our economy. In fact, this expression can be shown to deliver a lower bound on aggregate variance in the full-fledged model.

**Proposition 9.** *For large $N$, the variance is bounded below by:*

$$Var(g_t^a) \geq \sigma_\epsilon^2 H_t (1 + \gamma)^2 \tag{24}$$

# 4 Macro Evidence on Size Dispersion and Volatility

This section documents new stylized facts about the evolution of the joint distribution of firm size and firm-level volatility

## 4.1 Data

We conduct our analysis using stock market data from CRSP over the period 1926–2010 and cash flow data from CRSP/Compustat over 1952–2010. We consider market and fundamental measures of firm size and firm volatility calculated at the annual frequency. For size, we use equity market value at the end of the calendar, or total sales within the calendar year. Market volatility is defined as the standard deviation of daily stock returns during the calendar year. Fundamental volatility in year $t$ is defined as the standard deviation of quarterly sales growth (over the same quarter the previous year) within calendar years $t$ to $t + 4$. We also consider fundamental volatility measured by the standard deviation of quarterly sales growth within a single calendar year. The one and five year fundamental volatility estimates are qualitatively identical, though the one year measure is noisier because it uses only four observations.

Section C in the separate appendix establishes that log size and log volatility are approximately normally distributed in the data. The near-log normality of size and volatility distributions is convenient in that the dynamics of each distribution may be summarized with two time series: the cross section mean and standard deviation of the log quantities. We next examine these time series in detail.

## 4.2 Comovement of Size and Volatility Distributions

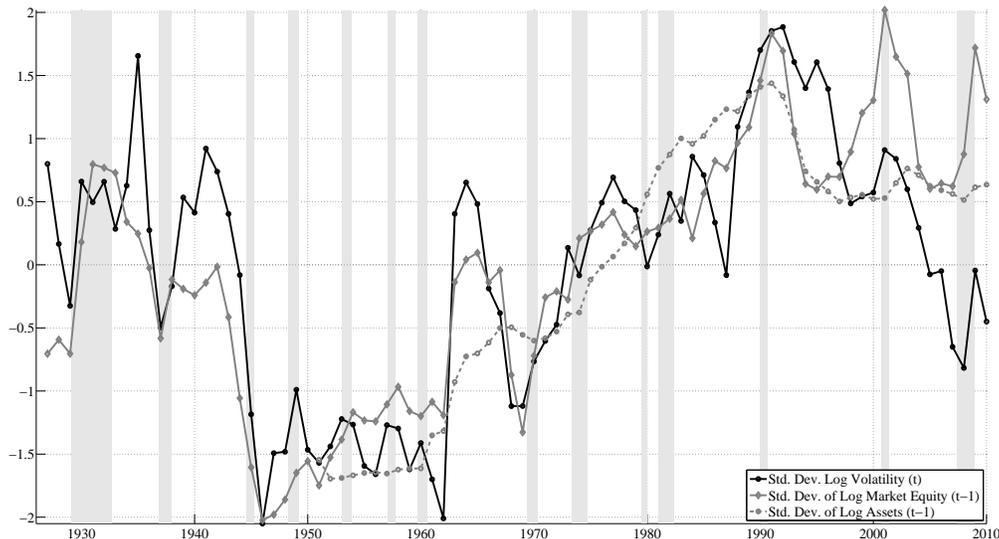Figure 1: AVERAGE VOLATILITY AND DISPERSION IN FIRM SIZE
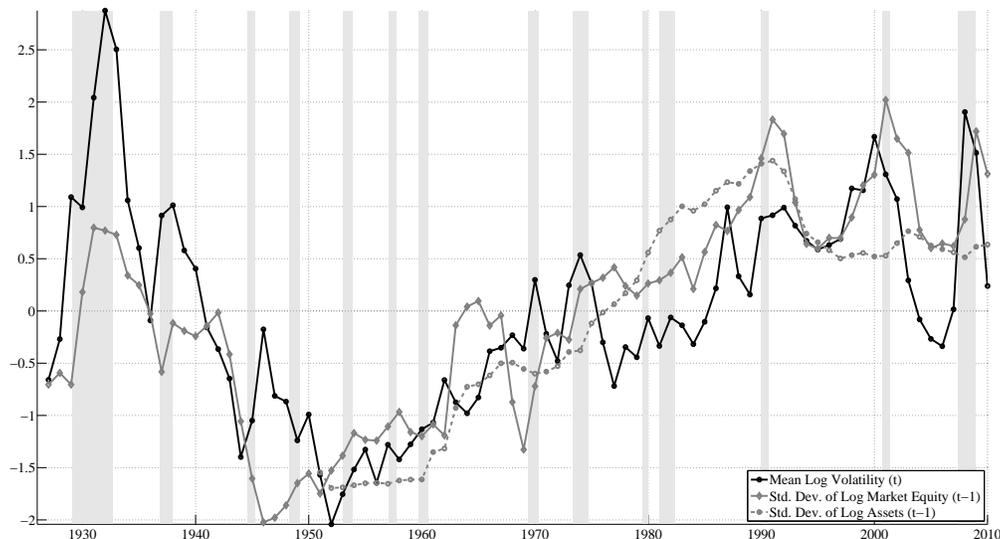


*Notes:* The figure plots cross section moments of the log size and log volatility distributions. All series standardized to have mean zero and variance one.

Figure 1 plots the cross-sectional average of log firm-level volatility against two different measures of size dispersion: the dispersion of log sales and the dispersion of log market equity across firms. The correlation between average volatility and the market-based measure of lagged market equity dispersion is 67%. The correlation between average firm volatility and the lagged sales dispersion is 61%.

Figure 2 plots the cross-sectional standard deviation of firm-level volatility against the same two measures of size dispersion. The correlation between volatility and the market-based measure of lagged size dispersion is 60%. The correlation between volatility and lagged sales dispersion is 41%.

Figure 3 plots the mean (top panel) and dispersion (bottom panel) of the log volatility distribution when volatilities are calculated either from returns are sales growth. Average return volatility and average sales growth volatility have an annual time series correlation of 64%, and their dispersions have a correlation of 49%, demonstrating a high degree of similarity between market volatilities and its (more coarsely measured) fundamental counterpart.

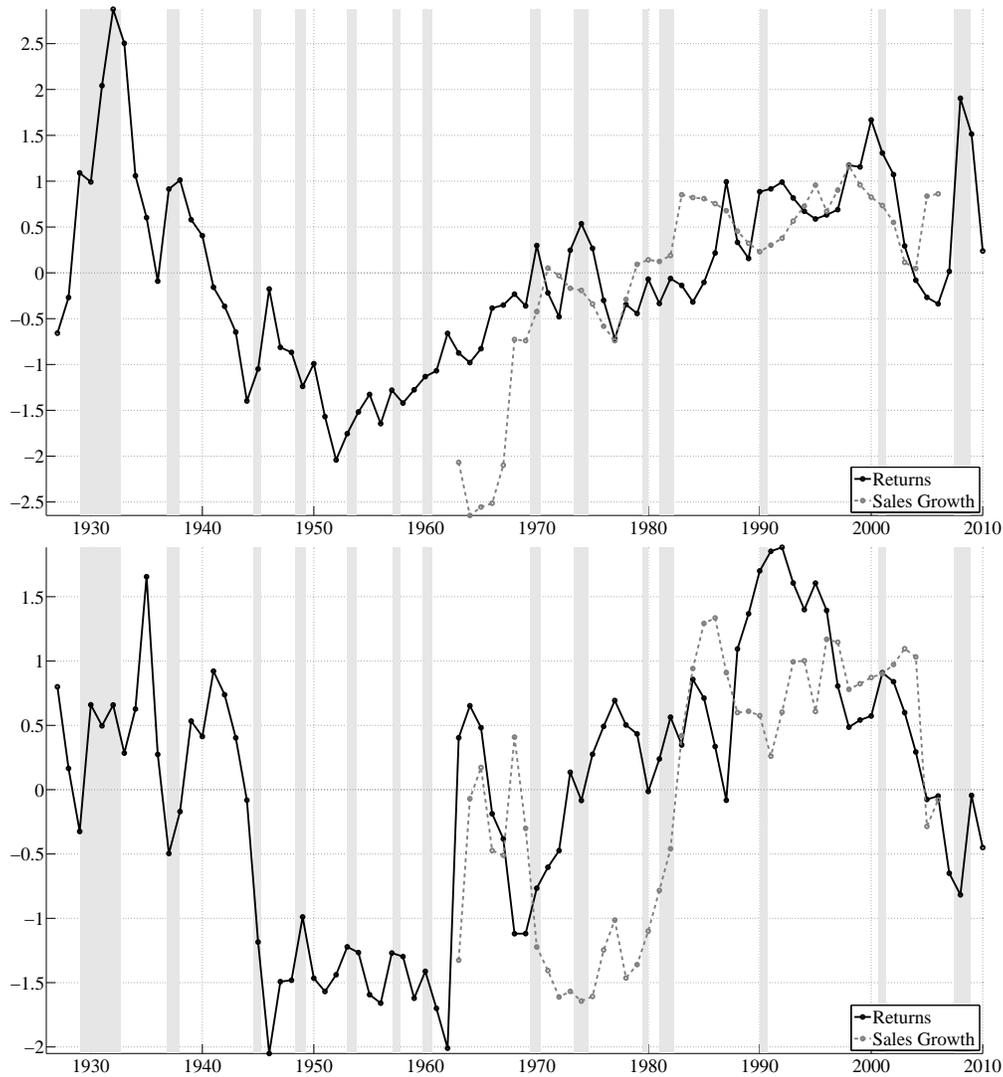Figure 2: DISPERSION IN VOLATILITY AND DISPERSION IN FIRM SIZE



*Notes:* The figure plots cross section moments of the log size and log volatility distributions. All series standardized to have mean zero and variance one.

It is important to note that these secular changes in the size distribution are not specific to publicly traded firms. In section F of the separate appendix, we document that the variance of the log employment distribution of all firms has also increased over the same sample. There secular decline in entry and exit rates over that sample may have contributed to the widening of the size distribution.

## 4.3 Granger Causality Tests

This evidence in Figures 1 and 2 raises a question of causality. Our network model predicts that movements in the size distribution cause the volatility distribution to change. When the size distribution spreads out (contracts) at time $t$, the network structure for the subsequent period adjusts, and diversification of growth rate shocks is hindered (enhanced). While we cannot address the question of economic causality in our empirical tests, we can test for causality in the time series sense. We use Granger causality tests to evaluate whether dispersion in firm sizes predicts the mean and standard deviation of the volatility distribution, after controlling for own lags of the dependent variable. Table 1 presents the results from

Figure 3: Fundamental and Market Volatility



*Notes:* The figure plots the cross section mean (top panel) and standard deviation (bottom panel) of the log volatility distribution for returns and sales growth. All series standardized to have mean zero and variance one.

these tests. We find that dispersion in the log market equity of US firms has statistically significant predictive power for average log firm return volatility, and dispersion in log firm volatility. The reverse is not true. After controlling for own lags of size moments, moments of the volatility distribution do not predict the size distribution.[9] Both of these facts are

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[9]Lagged dispersion in log volatility appears to Granger cause size dispersion, but the coefficient has the wrong sign. The hypothesis has the one-sided alternative that volatility dispersion *positively* predicts size

Table 1: GRANGER CAUSALITY TESTS

| Dependent Variable | | | Independent Variables | |
|---|---|---|---|---|
| | | Intercept | $\mu_{\sigma,t-1}$ | $\sigma_{S,t-1}$ |
| $\mu_{\sigma,t}$ | Coeff | -1.18 | 0.74 | 0.07 |
| | $t$-stat. | *-2.82* | *8.31* | *2.21* |
| $\sigma_{S,t}$ | Coeff | -0.28 | -0.11 | 0.97 |
| | $t$-stat. | *-0.47* | *-0.88* | *16.01* |
| | | Intercept | $\sigma_{\sigma,t-1}$ | $\sigma_{S,t-1}$ |
| $\sigma_{\sigma,t}$ | Coeff | -0.02 | 0.61 | 0.04 |
| | $t$-stat. | *-1.48* | *5.86* | *3.12* |
| $\sigma_{S,t}$ | Coeff | 0.20 | -1.19 | 1.04 |
| | $t$-stat. | *1.47* | *-2.25* | *15.88* |

*Notes:* Annual data 1926–2010. The table reports results of Granger causality tests for the ability of log firm size dispersion ($\sigma_{S,t-1}$) to predict the mean ($\mu_{\sigma,t}$) and standard deviation ($\sigma_{\sigma,t}$) of the log volatility distribution. We use the market based volatility measure constructed from stock returns and the market equity measure of size.

consistent with the implications of our network model. This evidence suggests that size dispersion leads the volatility distribution.

Table 2 shows that the strong correlation between moments of the volatility distribution and lagged size dispersion across markets, size quantiles and industries. We find that the the size distribution's predictive correlation with both mean volatility and dispersion in volatility is robust in each of these sample decompositions. Size dispersion predicts volatility moments among either NYSE or NASDAQ stocks, among large and small stocks, and within all industries.

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dispersion, thus this negative result leads us to fail to reject the null.

Table 2: Composition

| | # Firms | $\rho(\sigma_{\text{subset},s,t}, \sigma_{s,t})$ | $\rho(\mu_{\sigma,t}, \sigma_{s,t-1})$ | $\rho(\sigma_{\sigma,t}, \sigma_{s,t-1})$ |
|---|---|---|---|---|
| All stocks | 3004 | - | 71.7% | 79.3% |
| By sample period / exchange | | | | |
|   NYSE only | 1158 | 64.2% | 62.1% | 77.6% |
|   Non-NYSE | 3018 | 89.9% | 58.1% | 40.7% |
|   At least 50 yrs | 347 | 78.1% | 44.5% | 62.7% |
|   Random 500 | 500 | 90.5% | 64.9% | 80.7% |
| By size | | | | |
|   Smallest third | 1000 | 67.7% | 71.7% | 51.9% |
|   Middle third | 1000 | 87.7% | 61.6% | 69.8% |
|   Largest third | 1003 | 86.8% | 55.9% | 73.4% |
| By industry | | | | |
|   Consumer Non-Dur. | 248 | 91.4% | 64.4% | 72.3% |
|   Consumer Durables | 107 | 87.6% | 33.4% | 74.9% |
|   Manufacturing | 528 | 91.9% | 53.1% | 79.8% |
|   Energy | 140 | 72.9% | 68.4% | 67.9% |
|   Technology | 413 | 88.1% | 85.2% | 58.2% |
|   Telecom | 55 | 23.6% | 14.3% | 10.6% |
|   Retail | 310 | 86.5% | 69.0% | 69.8% |
|   Healthcare | 188 | 69.5% | 69.4% | 50.6% |
|   Utilities | 112 | 67.8% | 19.7% | 64.8% |
|   Other | 904 | 82.8% | 63.3% | 65.7% |

*Notes:* Annual data 1926-2010. We use the market based volatility measure constructed from stock returns and the market-based measure of size (market equity). The first column reports the time series average of the cross section number of firms. The second column reports the time-series correlation between the size dispersion of the sub-cross section and the dispersion of the full cross section. Column three reports the correlation between average log volatility ($\mu_{\sigma,t}$) and lagged log size dispersion ($\sigma_{S,t-1}$) and column four reports the correlation between dispersion in log volatility ($\sigma_{\sigma,t}$) and lagged log size dispersion.

Figure 4: Trend and Cycle in Size and Volatility Distributions
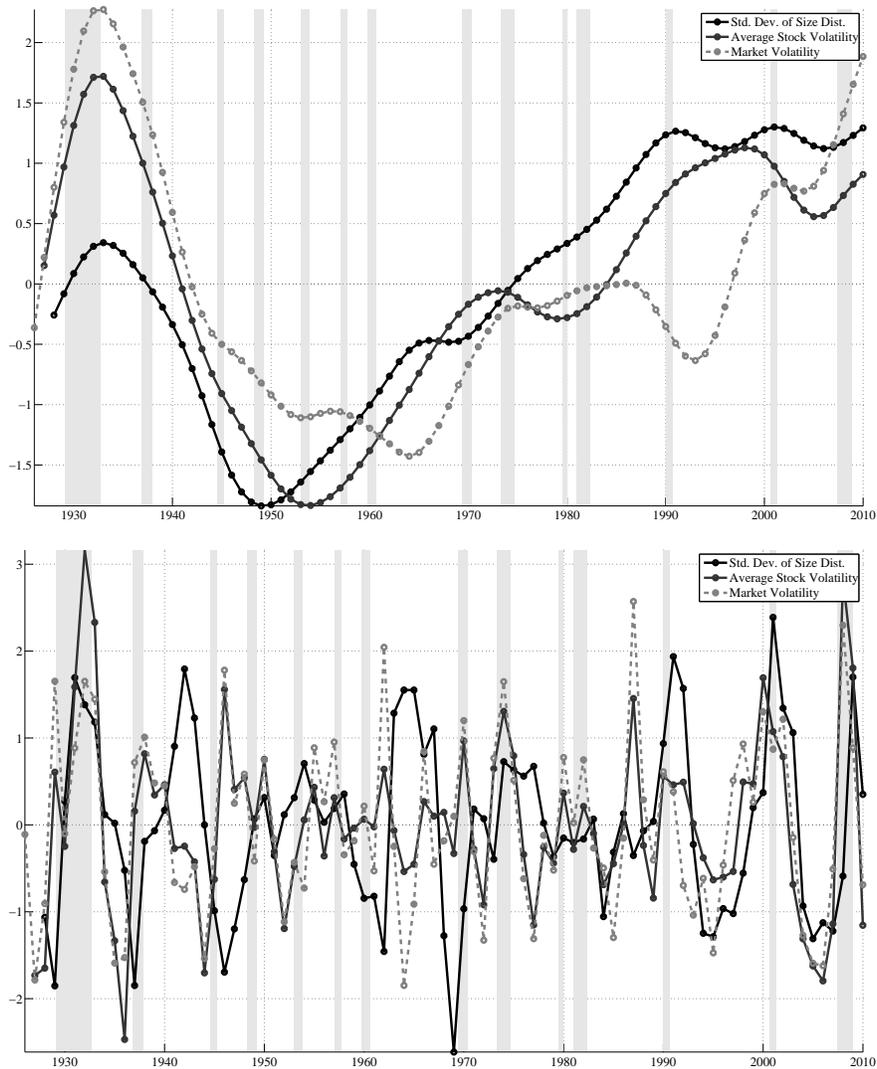


*Notes:* The figure plots HP-filtered trend and cycle components of time series moments of size and volatility distributions using smoothing parameter of 50.

## 4.4  Frequency Decomposition

To distinguish between the trend and cycle in the size and volatility moments, we apply the Hodrick-Prescott filter with a smoothing parameter of 50. The top panel shows the trend component. The bottom panel shows the cycle component. The top panel plots the trend in the cross-sectional average of log firm-level volatility against the cross-sectional dispersion of log market equity and log market volatility. The correlation between the trend components

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in average volatility and the market-based measure of lagged size dispersion is 90%. The correlation between lagged size dispersion and the trend component in market volatility is 55%.

The correlation between the cycle component in average log volatility and lagged market equity dispersion is 26%. These results suggest that the bulk of the predictive relation between the dispersion in the firm size distribution and moments of the volatility distribution occurs at lower frequencies. However, there is also significant evidence of correlation between cyclical volatility and size dispersion.

## 4.5   Volatility Factor Structure

Recent research has documented a puzzling degree of common variation in the panel of firm-level volatilities. Kelly, Lustig and Van Nieuwerburgh (2012) show that firm-level stock return volatilities share a single common factor that explains roughly 35% of the variation in log volatilities for the entire panel of CRSP stocks. This $R^2$ is nearly twice as high, roughly 70%, for the 100 Fama-French (1993) portfolios. They also show that this strong factor structure is not only a feature of return volatilities, but also holds for sales growth volatilities. The puzzling aspect of this result is that the factor structure remains nearly completely intact after removing all common variation in returns (or sales growth rates) by extracting principal components, so common volatility dynamics are unlikely to be driven by an omitted common factor.

The results of Section 3 suggest that our granular network model may be able to explain the puzzling comovement in firm volatility. (19) predicts an approximate factor structure among the volatilities of all firms, and suggests that concentration of the lagged economy-wide size distribution is the appropriate factor. Furthermore, Equation (33) predicts that factor model residuals will possess a similar degree of volatility comovement, despite residual growth rates themselves being nearly correlated.

In Table 3, we report results of panel volatility regressions for three different factor mod-

Table 3: $R^2$ of Volatility Factor Models

| | Factors | | |
| --- | --- | --- | --- |
| | $\sigma_{S,t-1}$ | $\mu_{\sigma,t-1}$ | $\mu_{\sigma,t}$ |
| Panel A: Return Volatility | | | |
| Total | 24.4 | 25.9 | 39.3 |
| Residual | 24.7 | 26.5 | 37.5 |
| | | | |
| Panel B: Sales Growth Volatility | | | |
| Total | 21.8 | 23.4 | 24.3 |
| Residual | 21.4 | 25.8 | 27.8 |

*Notes:* The table reports factor model estimates for the panel of firm-year volatility observations. In Panel A, total volatility is measured as standard deviation of daily returns within the calendar year, and residual volatility is estimated from daily regressions of firm returns on the value-weighted market portfolio within the calendar year. In Panel B, total volatility in year $t$ is measured as standard deviation of quarterly observations of year-on-year sales growth for each stock in calendar years $t$ to $t + 4$. Residual volatility is measured from regressions of firms sales growth on the sales-weighted average growth rate for all firms. All volatility factor regressions take the form $\log \sigma_{i,t} = a_i + b_i \text{factor}_t + e_{i,t}$. We consider three different volatility factors. The first, motivated by our network model, is the lagged cross section standard deviation of log market equity, $\sigma_{S,t-1}$. The second and third factors we consider are the lagged and contemporaneous cross section average log volatility, $\mu_{\sigma,t-1}$ and $\mu_{\sigma,t}$. We report the pooled factor model $R^2$ in percent.

els. The panels consist of volatility of total returns (or sales growth rates) and idiosyncratic volatilities. Idiosyncratic volatilities are calculated in a one factor model regression of stock returns on the value-weighted market return or, for sales growth, of individual growth rates on the sales-weighted aggregate growth rates. In both cases, residuals have average pairwise correlations that are below 2% in absolute value, despite the original returns having average correlations over 25% on average.

The first factor that we consider is the standard deviation of the lagged log market equity distribution, in line with Equation A. The second factor we consider is the sales-weighted average of the left hand side volatility variable. The cross section average is a natural benchmark for factor model comparison and it is typically highly correlated with the first principal component of the panel. When this factor is used, we estimate the model with the lagged average volatility to maintain comparable timing with the conditioning factor

Figure 5: AVERAGE FIRM VOLATILITY BY SIZE QUINTILE
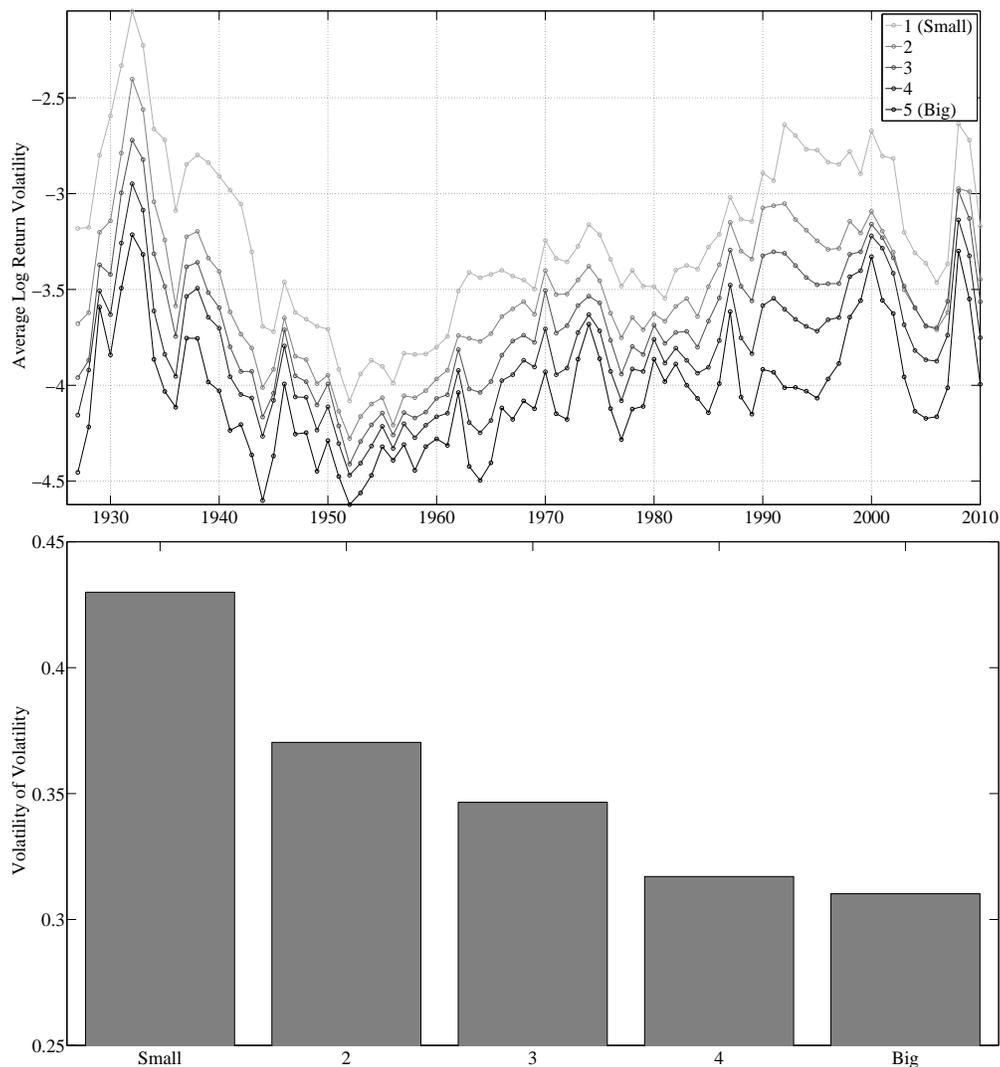


*Notes:* The top panel plots average log return volatility within CRSP market equity quintiles. The bottom panel reports time series standard deviation of average volatility within each quintile.

implied by our model, and also with the contemporaneous average volatility. We report the full panel $R^2$ based on each factor.

The table shows that the lagged size dispersion has the same degree of explanatory power as the lagged average volatility. Both factors capture about 25% of the panel variation in firm return volatility (Panel A). The contemporaneous average volatility can explain closer to 40% of the variation, but uses a finer conditioning information set. Panel B repeats

these regressions for the panel of total and idiosyncratic sales growth volatility. In this case the dispersion in firm sizes explains over 20% of the volatility panel variation, compared to around 25% for both contemporaneous and lagged average sales growth volatility.

The model also predicts that larger firms have lower loadings on the size dispersion factor, which lowers both the level of their volatilities relative to small firms, and also lowers their volatility of volatility. Figure 5 plot the average log firm volatility within size quintiles each year (top panel), and the volatility of average volatility for size quintiles (bottom panel). The figures show that large firms have lower levels of volatility and also less time series variation in volatility. Table 3 and Figure 5 are consistent with the model's prediction that dispersion in firm sizes predicts the entire panel of firm-level volatilities.

## 4.6 Aggregate Volatility Accounting

The divergence between firm-level and aggregate volatility measures has long puzzled financial economists. Candidate explanations can be divided into two classes, real and financial. Our stylized facts about the joint distribution of volatility and firm size largely survive even if we do not use market-based measures of volatility and/or size. This evidence largely rules out financial explanations of these stylized facts.

We find that accounting for changes in the size distribution nullifies the volatility trends identified in work by Campbell et al. (2001). Figure 6 plots the log of aggregate market volatility, as well as the residual from a regression of log market volatility on the cross-sectional dispersion in log market equity. This regression has an $R^2$ of 51.3%. The dotted lines show estimated time trends in the original volatility data (black) and in the residual (gray). The figure shows that, after controlling for fluctuations in the firm size distribution over time, there is no anomalous trend in average idiosyncratic volatility in the post-war era. This is consistent with the derivation of Section 3, which argues that the dispersion in firm size influences firm-level volatility. Admittedly, our model cannot account for the declining volatility of the market return over the same period, but this period decline with

Figure 6: Trends in Average Firm Volatility
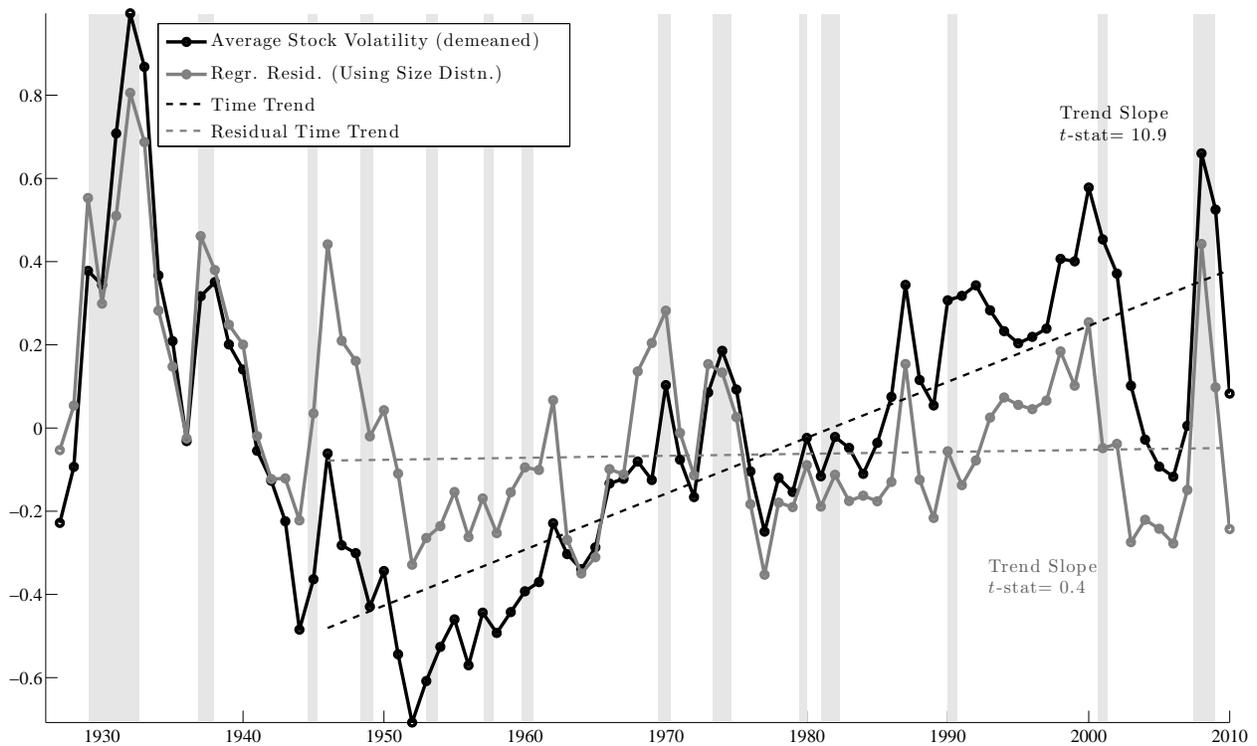


*Notes:* The figure plot log volatility of the value-weighted market equity portfolio (black line) as well as the residual from a regression of log market volatility on the cross-sectional dispersion in log market equity (gray line). It also shows estimated time trends in the original volatility data (black dotted line) and in the residual (gray dotted line), and reports the *t*-statistic for the time trend coefficient estimate.

a secular decline in the volatility of aggregate output consumption growth and hence is not too surprising.

# 5 Micro Evidence of Granular Networks

This section discusses the micro evidence on business networks.

## 5.1 Firm-level Network Data

Our data for annual firm-level linkages comes from the Compustat database. It includes the fraction of a firm's dollar sales to each of its major customers. Firms are required to

supply customer information in accordance with Financial Accounting Standards Rule No. 131, in which a major customer is defined as any firm that is responsible for more than 10% of the reporting sellers revenue. Firms have discretion in reporting relationships with customers that account for less than 10% of their sales, and this is occasionally observed. To compare the linkage data to our model, we strictly impose the 10% sales truncation in both the data and simulation when we calibrate the model. We discuss this further in the next section. Our data set covers the period 1980-2009, and includes 48,839 customer-supplier-year observations. Our data has been carefully linked to CRSP market equity data by Cohen and Frazzini (2008), which allows us to associate information on firms' network connectivity with their market equity size and their return volatility.[10]

The customer-supplier relationship data that we use is from Compustat, and is subject to two potential sources of bias. First, the data only include publicly traded firms. Second, suppliers are only obligated to report material customer relationships, including any customer that is responsible for more than 10% of the supplier's sales. We take two steps to address this empirical limitation. First, and most importantly, we simulate our model using the full firm size distribution (including private firms), and then censor the simulated data in a manner that matches the omission of small firms from Compustat data. Similarly, we censor all simulated customer-supplier links whose weights fall below 10% of the supplier's sales. By calibrating a censored simulation against censored data, we are able to draw inferences about network relationships among firms in the full, uncensored model economy. The second step we take to address censoring issues is to show that the size and volatility dynamics uncovered among public firms continues to hold when we pool public firm data with coarser data on private firms available from the Census Bureau (see Appendix F).

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[10]Our empirical work is related to Cohen and Frazzini (2008), who document that news about business partners does not immediately get reflected into prices. They use the same data on the business network of firms gathered from their annual reports. We use their business network data and we find that volatility is propagated upstream to suppliers as opposed to downstream, to customers. Atalay, Hortaçsu, Roberts, and Syverson (2011) use the same data to develop a realistic model of the buyer-supplier networks in the U.S. economy. We will use their insights to develop our model.

Table 4: OVERVIEW OF SIZE, VOLATILITY, AND CUSTOMER-SUPPLIER NETWORK STRUCTURE

| | Supplier | | | | | Customer | | | |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| | $Rk_i(w_{i,j,t})$ | $\log S_{i,t}$ | $\log S_{i,t}$ | $\log H_{i,t}^{out}$ | $\log H_{i,t}^{out}$ | $\log S_{i,t}$ | $\log S_{i,t}$ | $\log H_{i,t}^{in}$ | $\log H_{i,t}^{in}$ |
| | $Rk_i(S_{j,t})$ | Out-Deg. | $\log H_{i,t}^{out}$ | $\log \sigma_{i,t}$ (r) | $\log \sigma_{i,t}$ (s) | In-Deg. | $\log H_{i,t}^{in}$ | $\log \sigma_{i,t}$ (r) | $\log \sigma_{i,t}$ (s) |
| Average | 0.617 | 0.019 | -0.367 | 0.306 | 0.212 | 0.542 | -0.008 | 0.015 | -0.014 |
| Pooled | 0.708 | 0.069 | -0.350 | 0.314 | 0.201 | 0.534 | -0.014 | 0.018 | -0.013 |

*Notes:* The table reports annual cross section correlations between various features of firm's customer-supplier network (based on Compustat data) with firm's size and volatility. Annual data for 1980-2009.

## 5.2 Network Moments

Out-degrees range between one and 24, while in-degrees range from one to 130. We expect that both of these distributions are highly distorted due to the truncated nature of the data. Out-degrees can reach 24 since some suppliers (23%) voluntarily report customers that fall below the 10% sales threshold. The maximum out-degrees falls to 5 when we strictly impose the 10% sales truncation. Section E provides more details.

In Table 4, we report correlation statistics of the customer-supplier linkage data that are relevant to the model structure described in Section 2. We calculate cross section correlations for each yearly network realization in the Compustat linkage data. We also report the time series average of annual correlations and a pooled correlation for all firm-year observations combined. In our network data calculations, size is defined as firm's annual sales.

The model assumes that suppliers have stronger connections with their larger customers, and the table shows that this is indeed a very strong feature of the Compustat linkage data. For each supplier $i$, we calculate the rank correlation between its customer's total sales, and the fraction of $i$'s revenues that are due to each customer. We then average this number across all suppliers in a given year, this is reported in column (1). In an average year, a typical supplier has a rank correlation of 61.7% between the size of its customers and the fraction of total sales that each customer is responsible for. In the pooled supplier-year

panel, this correlation is 70.8%. This supports our assumption that linkage weight is closely related to the size of the customer.

We also assume that larger suppliers are connected to more customers on average. The data cannot speak directly to this assumption due to the truncation of all linkages whose weights fall below 10%. For example, a small firm that has a single customer accounting for 100% of its revenue can show up as having more links than a firm with 11 equally important customers (weights of 9.1%) due to truncation. Column (2) shows that any correlation that might exist between size and number of customers is largely destroyed by truncation, with an average annual correlation of 1.9% and a pooled correlation of 6.9%.

Column (3) reports the correlation between a supplier's size and the size Herfindahl of its customer network. As in the model, a supplier's customer Herfindahl is calculated based on the supplier's weighted out-degrees, $H_{t,i}^{out} = \sum_j w_{i,j,t}^2$. Due to the log normal feature of our model, we calculate correlations between log size and log Herfindahl. Our model predicts that larger suppliers are more diversified and thus have lower customer Herfindahls. The average and pooled correlations of -36.7% and -35.0% are consistent with this prediction.

Column (4) compares suppliers' log network Herfindahls to their log return volatilities calculated from CRSP data. The model suggests that well-diversified suppliers (those with lower Herfindahls) have lower volatility. This relationship is also identified in the linkage data, where Herfindahl and volatility (in logs) have an average correlation of 30.6% (or 31.4% when supplier-years are pooled). Column (5) repeats this calculation using quarterly sales growth volatility in place of return volatility and uncovers a similar association, with average and pooled correlations of 21.2% and 20.1%.

Whereas columns (1)-(5) focus on supplier networks, viewing the network from customers' perspectives is also informative for understanding network structure. In column (6) we calculate the correlation between customer size and in-degree. According to our model, the probability that a connection exists between supplier $i$ and customer $j$ depends only on $i$'s size and is independent of firm $j$'s size. Thus, our model would predict a zero correlation

between size and in-degree in the *absence* of link truncation. Because links with weights below 10% are not observed, and if larger customers are associated with higher linkage weights as we propose, then we would expect to see a strong association between customer size and in-degree in the truncated data. For instance, in our model, large firms like Apple or Walmart typically dominate their suppliers sales due to their massive sizes. While we assume smaller firms are equally likely to be customers as large firms, our model predicts that these weak links with small customers will not show up in the data, leaving large firms counted as customers far more often, and giving rise to the high correlation between size and in-degree that we see.

Column (7) shows that there is effectively zero association between a customer's in-degree Herfindahl, $H_{t,i}^{in} = \sum_j w_{j,i,t}^2$, and its size. The average and pooled correlations are -0.8% and -1.4%. Nor is there an association between customer's in-degree Herfindahl and customer volatility, as shown by correlations weaker than $\pm 1.8\%$ in columns (8) (return volatility) and (9) (sales growth volatility).

A comparison of results in columns (1)-(5), which summarize features of suppliers networks, to results on customer networks in columns (6)-(9) suggests an interesting interpretation of shock propagation through customer-supplier networks. First, results suggest that the strength of links depend strongly on customer size (column (1)) as in our model. Second, the association between a supplier's size and its number of customers (columns (2) and (6)) in the data is also consistent with our model assumption once the data truncation is taken into account. Third, suppliers' linkage concentration appears to associate strongly with suppliers' volatility, while this is not the case for customers. This is consistent with the notion that shocks are propagated upstream, from customers to suppliers, and that the effects of these shocks on firms' volatility depends on how well-diversified a firm is in its customer base.

Table 5: DETERMINANTS OF FIRM-LEVEL VOLATILITY

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Dependent Variable: Log Firm Volatility | | | | | | | | | |
| Log Sales | -0.16 | | -0.14 | -0.15 | | -0.12 | -0.14 | | -0.12 | -0.13 | | -0.12 | -0.13 | | -0.12 |
| | -16.71 | | -15.57 | -25.56 | | -23.93 | -19.32 | | -12.10 | -33.10 | | -19.93 | -20.61 | | -17.78 |
| Cust. Herf. | | 0.88 | 0.17 | | 0.63 | 0.17 | | 0.53 | 0.15 | | 0.55 | 0.13 | | 0.74 | 0.17 |
| | | 7.14 | 3.09 | | 8.74 | 3.43 | | 7.01 | 2.65 | | 8.69 | 3.27 | | 28.54 | 8.81 |
| Log Age | | | | -0.05 | -0.26 | -0.11 | -0.08 | -0.22 | -0.11 | | | | | | |
| | | | | -3.07 | -11.32 | -6.60 | -3.52 | -6.58 | -6.10 | | | | | | |
| Leverage | | | | 0.29 | 0.10 | 0.24 | 0.30 | 0.04 | 0.23 | 0.32 | 0.13 | 0.26 | | | |
| | | | | 3.32 | 1.08 | 3.10 | 4.53 | 0.48 | 3.45 | 4.71 | 1.87 | 3.95 | | | |
| Ind. Conc. | | | | 0.32 | 0.36 | 0.35 | 0.44 | 0.41 | 0.37 | 0.32 | 0.30 | 0.29 | | | |
| | | | | 1.51 | 1.32 | 1.62 | 2.13 | 1.35 | 1.70 | 2.06 | 1.45 | 1.81 | | | |
| Inst. Hldg. | | | | | | | -0.03 | -0.50 | -0.06 | | | | | | |
| | | | | | | | -0.39 | -5.09 | -0.61 | | | | | | |
| Constant | -3.91 | -3.52 | -3.78 | -3.92 | -3.03 | -3.64 | -3.80 | -2.93 | -3.60 | -4.01 | -3.64 | -3.88 | -3.86 | -3.51 | -3.75 |
| | -88.45 | -54.43 | -76.93 | -49.72 | -39.79 | -42.73 | -32.42 | -30.61 | -30.98 | -72.31 | -50.66 | -59.97 | -111.89 | -94.74 | -82.52 |
| FE | None | None | None | None | None | None | None | None | None | Cohort | Cohort | Cohort | Ind. | Ind. | Ind. |
| Obs. | 171,034 | 38,030 | 37,202 | 145,070 | 32,887 | 32,887 | 113,425 | 31,318 | 31,318 | 145,247 | 32,901 | 32,901 | 171,034 | 38,030 | 37,202 |
| Adj. $R^2$ | 0.329 | 0.048 | 0.361 | 0.347 | 0.228 | 0.402 | 0.354 | 0.281 | 0.413 | 0.405 | 0.307 | 0.448 | 0.386 | 0.218 | 0.425 |

*Notes:* The table reports panel regressions of firm's return volatility on a range of characteristics including size (log sales), customer network Herfindahl, age, book leverage, competition measured by industry size concentration, ownership composition measures as fraction of shares held by 13f-reporting institutions, as well as cohort and industry fixed effects in certain specifications.

## 5.3 Determinants of Firm-level Volatility

A large literature has examined the determinants of firm level volatility on the basis of firm characteristics, including Black (1976) who proposed that differences in leverage drive heterogeneity in firm volatility, Comin and Philippon (2006) who argue for industry competition and R&D intensity, Davis et al. (2007) who favor age effects, and Brandt et al. (2010) who argue that institiutional ownership is a key volatility driver. Our model predicts that there are two quantities that are equivalent in their determination of a firm's volatility – its size and its customer network concentration. Table 5 shows results of panel regressions of firm-level log annual return volatility on size and the customer-supplier network out-degree Herfindahl, while controlling for a range of firm characteristics including log age, leverage, industry concentration, institutional holdings, as well as industry and cohort fixed effects. Consistent with our model, we find that the two most important determinants of volatility are size and customer-size concentration. A 100% increase in the size of the firm decreases volatility by between 12% and 16%. An increase of customer Herfindahl from zero to one increases volatility by 88% without controlling for size; the effect is 17% when we control of size. Note that, in our model, size and network concentration are redundant since size determines network structure. Within our network model, a supplier's size and its customer Herfindahl are nearly perfectly correlated in the cross section. Given that concentration in the sales network is measured with substantial noise, it is highly likely that size captures an important part of the true concentration effect.

The results of this section are meant to motivate our network model. In Section 6 we analyze a complete calibration of our model to the Compustat linkage data and firm volatility data to draw broader conclusions regarding size and volatility in granular networks.

## 5.4 Industry-level Network Data

We have thus far presented our model and evaluated empirical networks at the firm level. As further corroborating evidence, we present evidence the industry-level input-output networks

appear largely consistent with the network structure that we have proposed. Industry input-output data are from the Bureau of Economic Analysis (BEA) at five year intervals between 1963 and 2002. Because industry definitions vary quite dramatically over time (with degree of disaggregation ranging form roughly 350 to 450 industries depending on the year), it is difficult to analyze network dynamics. However, this data is especially informative for evaluating cross section association between industry size and network structure since this data does not suffer the same truncation issue that we face in Compustat firm-level data.

Table 6: BEA INDUSTRY SIZE, VOLATILITY AND NETWORK STRUCTURE

| Correlation between ... | $\log S_{i,t}$ | $\log S_{i,t}$ | $\log S_{i,t}$ | $\log S_{i,t}$ |
| and ... | Out-Degree | $\log H_{i,t}$ | In-Degree | $\log H_{i,t}$ |
| Average | 0.505 | -0.179 | 0.569 | 0.235 |

*Notes:* Cross section correlations of industry size, out-degree, and out-Herfindahl from BEA industry benchmark input-output tables. The number of industries range between 300 and 500 depending on the year. Data for 1963, 1967, 1972,1977, 1982, 1987, 1992, 1997 and 2002.

Table 6 reports industry network correlations analogous to the firm network correlation in Table 4. We find the log total output of a supplier industry and the number of industries that it supplies to has an average correlation of 50.5% (ranging between 42.4% and 58.3% across years). Similarly, an industry's size has an average correlation of $-17.9\%$ with its purchaser Herfindahl (range of -5.3% to 29.0%). Both of these facts, that larger supplier industries have more connections and are better diversified, are consistent with our model. On the purchaser side, we find an average correlation of industry size with in-degree of 56.9%, and with supplier Herfindahl of 23.5%. Our model would not predict that a purchaser industry's in-degree would increase with size. But neither do we find evidence to suggest that purchaser industries are experiencing any diversification benefits from their supplier network, since larger industries have a *higher* in-degree concentration, not lower. On balance, we find the BEA industry network data to be consistent with the network structure we propose.

# 6 Joint Dynamics of the Size and Volatility Distribution

Our model generates interesting feedback from the size distribution to volatility. As the size dispersion increases, so does volatility, which in turn increases size dispersion.

## 6.1 Feedback from Volatility to Size Distribution

Suppose there is no minimum size requirement, no death and and no entry. The persistence parameter $\gamma$ governs the strength of the network effect. When $\gamma$ is zero, there are no network effects. This $\gamma = 0$ version of the model satisfies Gibrat's law: the mean growth rate and the variance of the growth rate are constant across firms, regardless of size, because the growth innovations are i.i.d. over time. In the absence of entry and exit, the size distribution is divergent because the standard deviation of the log size distribution $\sigma_{S,t} = \sigma_\varepsilon \sqrt{t}$. grows at a rate $\sqrt{t}$. This is true even if the size drift, $\mu_g$, is zero.[11]

The network's feedback effects contribute a term to the variance of the log size distribution that grows exponentially in $t$. Hence, $\sigma_{S,t}$ grows at a faster rate than $\sqrt{t}$, especially when $\gamma$ is large.

**Proposition 10.** *If the higher-order cumulants are positive, then the variance of the log size distribution at t is bounded below by:*

$$E[\sigma_{S,0}^2] + t\sigma_\epsilon^2 + \sigma_\epsilon^2 \gamma^2 Z \sum_{\tau=1}^{t} \exp\left[2\left(\tau\sigma_\epsilon^2 + \sigma_{S,0}^2\right)\right], t \geq 0$$

*Proof.* The inequality follows directly from the recursive expression for the variance of the log size distribution in Proposition 10. $\square$

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[11]The exogenous death rate and mimimim size requirement that we use are standard devices for ensuring the existence of a steady-state distribution, even in this $\gamma = 0$ version of the economy (see de Wit (2005)).

The network effects adds an exponential growth term to the cross-sectional variance of size. This term increases in $\gamma$. To discipline the behavior of the size distribution, we introduce entry and exit.[12]

## 6.2   Network Dynamics

To develop an understanding of the interaction between the size and volatility distribution in the full version of the network model, we introduce entry and exit into a version of the granular network model of Section 2 calibrated to match dynamics of the aggregate firm size and firm volatility distributions, as well as features of the customer-supplier network found in Compustat data.

The model begins at time 0 with an exogenous initial firm size distribution $\{S_{i,0}\}_{i=1}^{N}$, where each $S_{i,0}$ is drawn from a log normal distribution with mean $\mu_{S,0}$ and standard deviation $\sigma_{S,0}$. The linkage structure at time 0 is determined by the initial firm size distribution. Each element of the connections matrix, $\boldsymbol{B}_0 = [b_{i,j,0}]$, is drawn from a Bernoulli distribution with $P(b_{i,j,0} = 1) = p_{i,0}$ and

$$p_{i,0} = f_i(\{S_{i,0}\}_{i=1}^{N}) = S_{i,0} / \left( Z \sum_j (S_{j,0}) \right). \tag{25}$$

According to this rule, the probability that supplier $i$ and customer $j$ are connected depends linearly on the size of the supplier, not the customer. Furthermore, larger suppliers have more connections on average. This is the model's second size effect.

Firm sizes evolve according to the growth rate process (1), and linkages evolve with the size distribution over time. To induce stationarity in the model, we allow firms to die and be replaced by new firms, as in Atalay, Hortaçsu, Roberts, and Syverson (2011), whose

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[12]In the $\gamma = 0$ version of this model with a minimum size threshold, there is a steady-state size tail distribution that satisfies a power law ($G(S) = a/\tilde{S}^\zeta$ for some constant $a$ and $\zeta = 1$). Our model does not replicate this scale-independence in the tail probabilities when $\gamma > 0$. In fact, the exponent $\zeta(S)$ depends on size. When $S$ is very large, the exponent is close to 1, but it smaller for smaller sizes. This simply reflects the increase of the variance, which in turn produces more mixing of small and large firms. These results are derived in section G of the separate appendix.

model matches important dynamic features of the actual buyer-supplier network in the U.S. economy.

Each period $t$ arrives with a new an $N \times 1$ vector of firm sizes $\boldsymbol{S}_t$. A fraction $\delta$ of firms die randomly. Each defunct firm is replaced by a new firm that is drawn from the initial log normal firm size distribution (with parameters $\mu_{S,0}$ and $\sigma_{S,0}$).

The network of connections for surviving firms, governed by the $N \times N$ matrix $\boldsymbol{B}_t$, is redrawn on the basis of the prevailing size distribution. Connection probabilities each period are based on the size of suppliers as in (25). To induce persistence in links, we modify this rule, giving suppliers a relatively high probability of reconnecting to its customers from the previous period. The probability that a firm $i$ that was connected to firm $j$ in period $t-1$ is again connected to that same firm $j$ at time $t$ is given by

$$\max\{p_{i,j,t} + \kappa, 1\} \tag{26}$$

where $\kappa > 0$. Equation (26) preserves the probability rule $p_{i,t} = f_i(\{S_{i,t}\}_{i=1}^N)$ unconditional of past linkages.

These rules for the connection dynamics and firms' birth and death completely specify the network evolution and the growth rate process in (1).

## 6.3 Calibration

We simulate our model for $N = 2000$ firms and 1300 periods (years). We discard the first 300 observations to let the network settle down to its long run distribution, and compute our statistics by averaging over the last $T = 1000$ years. In each period we report moments based on the sample of the largest 1000 firms.[13] To further improve comparability between model and data, we compare the model results for a constant sample of 1000 firms to an

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[13]Focusing on the top 1000 firms is the model's way of recognizing that there are firms in the real world that are not measured in our Compustat sample (e.g., private firms), and that the firms that are measured are connected to these unmeasured firms affecting the former firms' network, size growth, and volatility. Our choice of $N = 2000$ is dictated by computational considerations.

appropriate corresponding Compustat subsample. We focus on the top-33%, a sample which contains 1,000 firms *on average* in the data. For completeness, we also report the empirical moments for the entire distribution of publicly traded firms (3000 firms on average).

Size in the data is measured as market capitalization deflated by the Consumer Price Index. In order to compare variance moments in model and data, we want to take into account that empirical variances are estimated with noise. [14]

In all simulations, we use the exact same random draws (and seed for the random number generator) to enhance comparability across models. All variation across models is thus produced by endogenously changing size and volatility distributions, not by random differences across simulations.

Because network data are truncated, as explained above, we implement the same truncation inside the model. That is, we set the weight $w_{ij,t}$ equal to zero whenever $w_{ij,t} < 0.10$.[15]

We choose the following paramers for our benchmark model. We set the mean exogenous firm growth rate $\mu$ equal to zero, which is the growth rate we observe for real market capitalization in the full cross-section.[16] The initial firm size distribution is log normal with mean $\mu_{S_0} = 10.20$ and standard deviation $\sigma_{S_0} = 1.06$. These numbers equal the time-series average of the observed cross-sectional mean and variance of the top-33% firm-size distribution.

The exogenous firm destruction rate $\delta$ is set to 5%, close to the time-series average firm exit rate in our sample of 4.2%. The probability of forming a connection between a

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[14]Therefore, we report "estimated" variance moments in the model, which we compute as:

$$\log\left(\widehat{V}ar_t[g_{t+1}]\right) = \log\left(Var_t[g_{t+1}]\right) + e_{t+1}$$

where $e \sim \mathcal{N}(0, \sigma_e^2)$ and $\sigma_e$ is the time series average of the cross-sectional standard deviation of $\log(Var_t[g_{t+1}])$.

[15]As in the actual data, our model-based moment calculations do *not* rescale the weights from our simulation to sum to one across customers $j$, so $\sum_j w_{ij,t} < 1$. As discussed above, SEC regulation requires that firms report all customers responsible for 10% or more of a supplier's sales. Some firms report more customers, however. We strictly impose the 10% truncation in the data by treating any relationship with reported sales weight below 10% as missing.

[16]The observed growth annual rate is 8% per year in the top of the firm size distribution, but this number is upwardly biased due to selection. Firms that leave the top of the size distribution are excluded from the growth calculations while firms that enter the top group are newly included.

supplier and a customer is governed by the function $p(\cdot)$. The scalar $Z$ governs the baseline likelihood of a connection for a firm $i$. We set $Z = 0.35$ which implies that the largest firm has a likelihood of connection of 27% on average. The parameter $\kappa$ gives the additional probability of a connection when firm $i$ and $j$ were already connected in the previous period. This additive term is not there for new firms (with no pre-existing connections). Thus, $\kappa$ governs the persistence of connections. We set $\kappa$ equal to 0.5 to match the observed 54% time-series average death rate of truncated links. In our benchmark model the average death rate of truncated links is 57% whereas the average death rate of untruncated links is 46%.

The two key parameters are $\gamma$, which governs the persistence of the network, and $\sigma$, which is the fundamental shock volatility. The parameter $\gamma$ governs how important shocks to customers' growth are to a supplier. The closer $\gamma$ is to one, the more important are higher-degree (indirect) effects, i.e., effects beyond the direct supplier-customer relationship. These two parameters are set to match as best as possible the mean and dispersion of firm volatility.

## 6.4 Calibration Targets

This section documents features of of the size, volatility and in/out-degree distribution of U.S. firms. We will make use of these moments to develop a calibrated version of our network economy.

### 6.4.1 Size Distribution Moments

Table 7 reports moments of the cross-sectional log size distribution $(\log S)$ in Panel A and the firm growth distribution $(g_t = \log S_t / \log S_{t-1})$ in Panel B. All reported moments are time-series averages unless explicitly mentioned otherwise. The first column reports moments for the full cross-section of firms observed in Compustat. The second column reports results for the top-33% of Compustat firms in each year. Column 2 is the main column of interest in the data. For example, 11.63 is the time-series average of the average log size of the

Table 7: FIRM SIZE DISTRIBUTION

|  | All | Top-33% | Model |
|---|---|---|---|
| **Panel A: Cross-sectional Moments of Log Size** | | | |
| Avg | 9.61 | 11.63 | 11.63 |
| SD | 1.79 | 1.06 | 1.07 |
| 5% | 6.86 | 10.38 | 10.32 |
| 10% | 7.39 | 10.47 | 10.44 |
| 25% | 8.33 | 10.78 | 10.79 |
| Med | 9.48 | 11.39 | 11.42 |
| 75% | 10.77 | 12.25 | 12.25 |
| 90% | 12.05 | 13.10 | 13.10 |
| 95% | 12.76 | 13.64 | 13.68 |
| **Panel B: Time Series Properties of Size Distribution** | | | |
| SD of $\sigma_{S,t}$ | 0.24 | 0.14 | 0.19 |
| AR(1) $\sigma_{S,t}$ | 0.947 | 0.939 | 0.995 |
| SD of $\sigma_{g,t}$ | 0.15 | 0.11 | 0.05 |
| $H_t$ | 0.010 | 0.012 | 0.020 |

*Notes:* All reported moments in Panels A and B are time-series averages of the listed year-by-year cross sectional moments (cross section average, standard deviation, and percentiles) for the sample 1926–2010. The first column reports the full cross-section of firms . The second column reports results for the top-33% of firms in each year. Column 3 reports the corresponding moments for the benchmark model (M1). Column 4 shows the first-degree approximation of the benchmark model (M2). Columns 5-7 show three variations of the full-degree network model for different parameter values, described in the main text, and labeled M2-M5. Panel A reports moments of the log size distribution, where size is defined in the data as market equity. Panel B reports the time-series standard deviation and time-series persistence of size dispersion $\sigma_{S,t}$, defined as the cross-sectional standard deviation of log size, the time-series standard deviation of the cross section standard deviation of log growth rates $\sigma_{g,t}$ and the time-series average of the economy-wide Herfindal index $H_t$.

large-firm sample, which is naturally higher than the 9.61 for the full Compustat sample. The dispersion of the log size distribution is 1.06 for the top-33% group and 1.79 for the full sample.

Panel B reports *aggregate* moments of the size distribution. The first two rows show that size dispersion (the cross-sectional standard deviation of log size) has high variability over time and is highly persistent. The time series standard deviation is 14% for the top-33% firms and 24% for all firms. The third row shows that the cross-sectional standard deviation of size *growth* (log size changes) is also volatile over time. The time series standard deviation is 15% for all firms and 11% for the top-33% firms. These results confirm that the size distribution moves around considerably over time. Finally, the size Herfindahl of the entire economy, a standard concentration measure, is 0.012 in the data.

### 6.4.2 Volatility Distribution

Table 8 reports moments of the cross-sectional log variance distribution. In columns 1 and 2, we compute the variance as the realized variance of daily stock returns within the year, which is then annualized (multiplied by 252), and logged. In columns 3 and 4, we compute the variance based on twenty quarters worth of year-over-year growth rates in log sales. The window includes four growth rates in the current year and 16 in the next four years. Because volatilities (standard deviations in levels) are more intuitive than log variances, we exponentiate the moments of log variance and then take their square root, which is what is reported in Table 8. All reported moments are time-series averages unless explicitly mentioned otherwise.

Panel A shows that average firm return-based volatility is 41% per year for the full cross-section and 30% per year for the large-firm sample. Average sales-based volatility is 29% for the full sample and 24% for the large-firm sample. The range of return-based (sales-based) volatilities goes from 17% (8%) at the fifth percentile to 54% (77%) at the 95th percentile for the latter group. The cross-sectional dispersion in return volatility is 73% in Column

2 and 97% in the full cross-section reported in Column 1. Sales-based volatility has even larger dispersion of 142% and 151%, in Columns (4) and (3) respectively.

Panel B shows moments of the joint cross-sectional distribution of firm size and firm volatility. The first row computes the cross-sectional correlation between log size at time $t$ and log variance at time $t + 1$, for each $t$ and then reports the time-series average. The second row reports the slope coefficient (beta) of a cross-sectional regression of log variance at time $t + 1$ on a constant and log size at time $t$; it reports the time-series average of that slope. Both are strongly negative in the data showing that large firms have lower volatility over the next period.

Panel C reports aggregate moments of the the volatility distribution and the joint size-volatility distribution. The first row shows the time-series standard deviation of average volatility: 69% in the full sample compared to a mean of 41% and 64% in the top-33% sample compared to a mean of 30%. Average volatility is highly variable over time. The second row shows that the cross-sectional standard deviation of log variance moves substantially over time in both samples and for both ways of measuring volatility. The time series standard deviation is 18% in the full sample and 12% in the top 1000 sample; both are lower than the time-series volatility of mean volatility.

The third row reports the time-series correlation between size dispersion at $t$ (the cross-sectional standard deviation of log size) and mean volatility (the cross-sectional mean of log variance) at time $t+1$. The two are strongly positively correlated in the data: 0.72 in the full sample and 0.52 in the top sample. Similarly, size dispersion is strongly positively correlated with volatility dispersion (the cross-sectional standard deviation of log variance) over time: 0.79 and 0.72, in Columns 1 and 2 respectively.

The last three rows investigate the cyclicality of moments of the size- and volatility distribution. We compute aggregate size growth, measured as the weighted average growth rate between $t$ and $t + 1$ with weights measured at $t$ that are equal to the relative size of a firm. We correlate this aggregate growth rate with size dispersion at $t$ (the cross-sectional

standard deviation of log size) and with the mean and the standard deviation of volatility at $t+1$ (the cross-sectional mean and standard deviation of log variance). Size dispersion shows mild counter-cyclicality. Return-based average volatility seems counter-cyclical, and so does the dispersion of volatility in the top group of firms.[17] This is consistent with the new findings reported by Bloom, Floetotto, Jaimovich, and Terry (2012).

### 6.4.3 Network Moments

This section directly examines the connection between the properties of the customer-supplier network and firm-level volatility. To help us understand the customer-supplier network in the U.S. economy, we use the data on economic linkages from Compustat (Cohen and Frazzini 2008). The data covers a subset of of CRSP/Compustat firms observed from 1980 until 2009. Firms are required to report customers representing more than 10% of reported sales.

We report several moments that directly pertain to the network in Table 9. All moments are time-series averages. We report the median and $99^{th}$ percentile of the cross-sectional distribution of the number of out-degrees $K^{out}$, the number of customers that a supplier is connected to, and the number of in-degrees $K^{in}$, the number of suppliers a customer is connected to. We also report the median and $99^{th}$ percentile of the cross-sectional distribution of the out- and in-degree Herfindahl indices $H^{out}$ and $H^{in}$. The latter indices for are defined as follows for a given firm $i$ :

$$H_{t,i}^{out} = \sum_{j \in \mathcal{C}_i} w_{i,j,t}^2$$
$$H_{t,i}^{in} = \sum_{j \in \mathcal{S}_i} \tilde{w}_{i,j,t}^2$$

where $\mathcal{C}_i$ denotes the set of customers of firm $i$ and $\mathcal{S}_i$ is the set of suppliers of firm $i$. The $\tilde{w}$ are elements of a matrix that measures the strength of a given customer's suppliers; the

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[17]Sales-based volatility shows little covariance with aggregate growth in the economy, but this could be because the volatility is computed from observations that span five years of data.

Table 8: FIRM VOLATILITY DISTRIBUTION

| | All Returns | Top-33% Returns | All Sales | Top-33% Sales | Model |
|---|---|---|---|---|---|
| **Panel A: Cross-Sectional Moments of Firm Volatility** | | | | | |
| Avg | 0.40 | 0.30 | 0.29 | 0.24 | 0.37 |
| SD | 0.96 | 0.73 | 1.51 | 1.42 | 0.45 |
| 5% | 0.19 | 0.17 | 0.09 | 0.08 | 0.26 |
| 10% | 0.22 | 0.19 | 0.11 | 0.10 | 0.28 |
| 25% | 0.29 | 0.24 | 0.17 | 0.14 | 0.31 |
| Med | 0.40 | 0.30 | 0.29 | 0.23 | 0.36 |
| 75% | 0.56 | 0.38 | 0.48 | 0.38 | 0.42 |
| 90% | 0.75 | 0.48 | 0.77 | 0.60 | 0.49 |
| 95% | 0.90 | 0.54 | 1.05 | 0.77 | 0.54 |
| **Panel B: Cross-Sectional Moments of Size-Volatility Distribution** | | | | | |
| $Corr(S_t, V_{t+1})$ | $-0.57$ | $-0.33$ | $-0.33$ | $-0.18$ | $-0.42$ |
| $\beta(S_t, V_{t+1})$ | $-0.32$ | $-0.23$ | $-0.27$ | $-0.22$ | $-0.18$ |
| **Panel C: Time Series Properties of Volatility Distribution** | | | | | |
| SD of $\mu_{\sigma^2,t}$ | 0.69 | 0.64 | 0.46 | 0.42 | 0.20 |
| SD of $\sigma_{\sigma^2,t}$ | 0.18 | 0.12 | 0.13 | 0.13 | 0.05 |
| $Corr(\sigma_{S,t}, \mu_{\sigma^2,t})$ | 0.71 | 0.55 | 0.51 | 0.51 | 0.95 |
| $Corr(\sigma_{S,t}, \sigma_{\sigma^2,t})$ | 0.76 | 0.76 | 0.57 | 0.38 | 0.77 |
| SD of $g_{agg,t}$ | 0.21 | 0.20 | 0.21 | 0.20 | 0.20 |
| $Corr(g_{agg,t}, \sigma_{S,t})$ | $-0.06$ | $-0.08$ | $-0.06$ | $-0.08$ | $-0.02$ |
| $Corr(g_{agg,t}, \mu_{\sigma^2,t})$ | $-0.32$ | $-0.43$ | 0.01 | $-0.00$ | $-0.01$ |
| $Corr(g_{agg,t}, \sigma_{\sigma^2,t})$ | 0.04 | $-0.19$ | 0.18 | 0.14 | 0.02 |

*Notes:* All reported moments in Panels A and B are time-series averages of the listed year-by-year cross sectional moments (cross section average, standard deviation, and percentiles) for the sample 1926–2010. The first and third columns report return volatility and sales growth volatility, respectively, for the full cross-section of firms. The second and fourth columns report results for the largest 33% of firms by market capitalization in each year. Annual variances in Columns 1 and 2 are calculated from daily stock return data, while variances in columns 3 and 4 are based on 20 quarters of sales growth rates (in the current and the next four years). Moments are computed based on the log variance distribution, but are expressed as volatilities in levels for exposition. That is, we exponentiate each moment of the log distribution and take the square root. Column 5 reports the corresponding moments for the benchmark model (M1). Column 6 shows the first-degree approximation of the benchmark model (M2). Columns 7-9 show three variations of the full-degree network model for different parameter values, described in the main text. Log variances in the model are constructed with added estimation noise to make them comparable to the moments in the data.

sum of the $\tilde{w}$ across suppliers is one. In the data, we only observe connections measured from the perspective of the supplier, and only those that represent more than 10% of the sales.[18] As a result, the observed out-degree distribution is a (potentially severely) truncated version of the true out-degree distribution. The problem is less pronounced for the in-degree distribution, since the truncation occurs based on supplier-reported data, but still highly relevant.

We find a median (truncated) out-degree of 1 and (truncated) $99^{th}$ percentile of 3.3 connections. The median (truncated) in-degree is 1 with a $99^{th}$ percentile of 17 or 32 connections depending on the sample. The Herfindahl indices, which are also based on truncated degree information, are less biased because large customers receive a large weight and are more likely to be in the database. We calculate cross-sectional correlations of in- and out-degrees and in- and out-Herfindahls at time $t$ with log size at $t$ and log variance at $t + 1$. We find a strong negative relationship between supplier size and out-Herfindahl; the correlation is -0.30. Large firms have a better-diversified portfolio of customers, and hence small out-Herfindahls. We also find a positive relationship between supplier volatility and out-Herfindahl: the correlation is 0.17. Firms with a more diversified portfolio of customers and smaller Herfindahls have lower volatility because they more effectively diversify the shocks that hit their customers. This also explains the negative correlation between firm size and firm volatility in Panel B of Table 8. We find a nearly zero correlation between truncated out-degree and size, but argue based on the model that truncation generates a downward bias in this correlation. The data suggest a positive 0.37 correlation between in-degree and size, but we argue based on the model that truncation induces an upward bias in this correlation. This provides evidence that the likelihood of a connection between a supplier and a customer does not strongly depend on the customer's size. We find a zero (modest positive) correlation between in-Herfindahl on the one hand and volatility (size) on the other hand.

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[18]Since reporting is voluntary for customers that represent less than 10% of sales, the data contain some shares smaller than 10%. We replace those by zeroes. In the model we will apply the same truncation.

Table 9: NETWORK MOMENTS

| | All Returns | Top-33% Returns | All Sales | Top-33% Sales | Model |
|---|---|---|---|---|---|
| **Panel A: Out-degree Moments** | | | | | |
| median $K^{out}$ | 1.00 | 1.00 | – | – | 1.71 |
| $99^{th}$ % $K^{out}$ | 3.38 | 3.32 | – | – | 4.66 |
| median $H^{out}$ | 0.05 | 0.04 | – | – | 0.60 |
| $99^{th}$ % $H^{out}$ | 0.95 | 0.54 | – | – | 1.00 |
| $Corr(K_t^{out}, S_t)$ | 0.01 | −0.07 | – | – | 0.34 |
| $Corr(H_t^{out}, S_t)$ | −0.30 | −0.09 | – | – | −0.71 |
| $Corr(H_t^{out}, V_{t+1})$ | 0.17 | 0.11 | 0.25 | 0.09 | 0.54 |
| **Panel B: In-degree Moments** | | | | | |
| median $K^{in}$ | 1.00 | 1.12 | – | – | 1.90 |
| $99^{th}$ % $K^{in}$ | 31.61 | 16.80 | – | – | 7.39 |
| median $H^{in}$ | 0.00 | 0.00 | – | – | 0.11 |
| $99^{th}$ % $H^{in}$ | 0.28 | 0.24 | – | – | 0.85 |
| $Corr(K_t^{in}, S_t)$ | 0.37 | 0.49 | – | – | 0.49 |
| $Corr(H_t^{in}, S_t)$ | −0.44 | −0.28 | – | – | 0.03 |
| $Corr(H_t^{in}, V_{t+1})$ | 0.25 | 0.19 | 0.09 | 0.12 | 0.01 |

*Notes:* All reported moments are time-series averages unless explicitly mentioned otherwise. The first and second columns report data for the cross-section of firms for which we have customer information from Compustat. The sample is 1980-2009. The variance in Column 1 is calculated based on daily stock return data, while the variance in Column 2 is based on 20 quarters worth of annual sales growth (in the current and the next four years). Column 3 reports the corresponding moments for the benchmark model (M1). Column 4 shows the first-degree approximation of the benchmark model (M2). Columns 5-7 show three variations of the full-degree network model for different parameter values, described in the main text. Log variances in the model are constructed with added estimation noise to make them comparable to the moments in the data.

## 6.5 Simulation Results

This section reports the simulation results. We focus on the benchmark calibration and leave a detailed discussion of the alternative calibrations for the appendix.

### 6.5.1 Benchmark Calibration

As can be seen from column M1 in Table 7, the benchmark model matches the main moments of the firm size distribution. It matches average firm size exactly. It also generates a large amount of cross-sectional heterogeneity in firm size. Column 3 of Table 7 shows that size dispersion averages 1.07, close to the observed 1.06 value reported in column 2. The inter-quartile range for log firm size is [10.79,12.25], matching the [10.78,12.25] range in the top-33% sample. At both extremes of the distribution, the model also compares favorably to the data.

Panel B shows that size dispersion moves around dramatically over time in the model and is very persistent. The model produces the same high time-series variability in the size dispersion as observed (0.19 compared to 0.14 in column 2 and 0.24 in column 1), but not that of the dispersion in size growth rates (0.05 versus 0.11 in column 2 and 0.15 in column 1). The benchmark model slightly overstates firm concentration: the economy-wide Herfindahl index is 0.020 compared to 0.012 in the data.

Next, we turn to the volatility distribution. Column 3 of Table 8 shows that the model generates high average firm volatility of 37%, in between the return volatility in the full sample of 40% and that in the top-33% sample of 30%. Average sales-based volatility in the data is lower at 29% and 24%, respectively. The model is capable of generating a wide range of volatility outcomes. The cross-sectional dispersion in volatility is 0.45, compared to 0.73 for the return-based measure (column 2), and 1.42 for the sales-based measure. The model generates the right volatility at the 90 and $95^{th}$ percentiles. The least volatile firms have a volatility of 26% in the model, but only 17% in the return data.

Panel B shows that the model generates a -42% correlation between size and volatility

at the firm level, similar to the -33% in the data. That relationship between log variance and log size is similar to the one in the data (slope coefficient of -0.18 versus -0.23). Panel C shows that the model generates substantial variability over time in both the cross-sectional mean of firm variance and in its dispersion, but still under-predicts these moments compared to the data. The model generates a high volatility of aggregate growth rates of 20%, close to the 20% in the data.

In the time-series, size dispersion is strongly positively correlated with both the mean of the log firm variance distribution (0.95 correlation) and its dispersion (0.77 correlation). These correlations are high in the data as well (0.55 and 0.76). These correlations are some of the key moments we focus on in this paper.

Table 9 shows the network-related properties of the model. Like in the data, the model generates a small median number of supplier-customer relationships after truncation: 1.71 on average in the model. The $99^{th}$ percentile of the truncated out-degree distribution is 4.66 in the model and 3.19 in the data. Truncation severely affects the number of out-degrees. The median of the *untruncated* $K^{out}$ distribution in the model is 3.71 and the $99^{th}$ percentile is 112. In the model, larger firms have more connections. The cross-sectional correlation between the truncated out-degree and log size is 0.34, which is 0.26 lower than the 0.60 correlation between the *untruncated* out-degree and log size. This downward bias means that the observed correlation between truncated out-degree and log size in the data (0.00) is downward biased. Applying the model-generated bias, the data would show a positive correlation between *untruncated* out-degree and log size (albeit a smaller one as in the model). That is, larger suppliers have more connections.

A reverse bias occurs for the correlation between in-degree and log size. The observed value of 0.26 is somewhat smaller than the 0.49 value in the benchmark model, where both numbers are based on the truncated in-degree distribution. However, the correlation between *untruncated* out-degree and log size 0.04 in the model, which implies that this moments is severely upward biased due to truncation. The magnitude of the bias is such that the

(unobserved) correlation between untruncated in-degree and log size could easily be zero. That zero correlation between customer size and number of connections is an important assumption of the model.

The model matches some properties of Herfindahls, but not others. The main shortcoming of the benchmark calibration is that its truncated out-degree Herfindahl is too high. The median is 0.60, much higher than the 0.05 value we observe in the data. The reason is that the number of connections a typical supplier has is small (the median out-degree is 3.7), and that large customers receive a large weight ($w_{ij}$ is strongly increasing in size of the supplier $S_j$).[19] On the positive side, the model generates a strong negative correlation between out-degree Herfindahl and size and a positive correlation between out-degree Herfindahl and volatility. Both are important features of the data. The model also matches the correlation pattern between in-Herfindahls and size and volatility. In the model, truncated and untruncated Herfindahls are very similar, suggesting no evidence for bias in the empirical counterparts.

Finally, we re-estimate the volatility factor regressions of Table 3. The corresponding $R^2$s for total volatility from left to right are 36%, 37%, and 40%, quite comparable to the data. Thus, the model replicates the strong factor structure in volatilities.

### 6.5.2   Alternative Calibrations

We compute several other models that illustrate various features of the model. These are discussed at length in the separate appendix in section H. These alternative model features show that strong network persistence is important (high $\gamma$, and a model with higher-order network effects). They also show that we can match the low observed out-Herfindahls by changing the parameters that govern the number and the importance of connections, without jeopardizing the main predictions of the model. That calibration, however, cannot generate enough firms with high enough volatility.

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[19]Model 5 partially remedies these issues.

# 7    Conclusion

We document new features of the joint evolution of the firm size and firm volatility distribution and propose a new model to account for these features. In the model, shocks are transmitted upstream from customers to suppliers. Firms sell products to an imperfectly diversified portfolio of customers. The larger the supplier, the more customer connections the supplier has, the better diversified it is and the lower its volatility. Large customers have a relatively strong influence on their suppliers, so shocks to large firms have an important effect throughout the economy. When the size dispersion of the economy increases, such large firms become more important and many firms' basket of customers becomes less diversified. In those times, average firm volatility is higher as is the cross-sectional dispersion of volatility. We provide direct evidence of such linkages, and use the supplier-customer relationship data to calibrate our model.

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# A  Proofs

- Proof of Proposition 1:

*Proof.* We will derive a lower bound on the average variance. We use the following notation to denote the matrix inverse:

$$\boldsymbol{v} = (\boldsymbol{I} - \gamma \boldsymbol{W}_t)^{-1} \quad = \quad \boldsymbol{I} + \gamma \boldsymbol{W}_t + \gamma^2 \boldsymbol{W}_t^2 + \gamma^3 \boldsymbol{W}_t^3 +$$

The average variance is given by $\frac{1}{N} \sigma_\epsilon^2 \sum_i \boldsymbol{e}_i' \boldsymbol{v} \boldsymbol{v}' \boldsymbol{e}_i'$. First, note that:

$$\boldsymbol{e}_i' \boldsymbol{v} \geq \boldsymbol{e}_i' (\boldsymbol{I} + \gamma \boldsymbol{W}_t),$$

because the $\boldsymbol{W}$ matrix only has positive entries. Hence, in turn, it follows that:

$$\boldsymbol{e}_i' \boldsymbol{v} \boldsymbol{v}' \boldsymbol{e}_i' \geq \boldsymbol{e}_i' (\boldsymbol{I} + \gamma \boldsymbol{W}_t)(\boldsymbol{I} + \gamma \boldsymbol{W}_t') \boldsymbol{e}_i'$$

We have derived the following lower bound on the average variance:

$$\frac{1}{N} \sum_i V_t (g_{i,t+1}) \geq \sigma_\epsilon^2 \left( 1 + \frac{2}{N} \gamma \sum_{i=1}^N w_{ii} + \frac{1}{N} \gamma^2 \sum_{i=1}^N \sum_{j=1}^N w_{ij,t}^2 \right)$$

We can restate this expression using the Herfindahl:

$$\frac{1}{N} \sum_i V_t (g_{i,t+1}) \geq \sigma_\varepsilon^2 \left( 1 + \gamma^2 \frac{1}{N} \sum_i H_{i,t} \right)$$

$\square$

- Proof of Proposition 3

*Proof.* In such a network, each firm $i$ only has one connection:

$$g_i = \mu_g + \gamma g_{i+1} + \varepsilon_i, \tag{27}$$

where firm $N+1$ is firm 1 by convention. Without loss of generality, this $\boldsymbol{W}_t$ matrix can be represented with ones above the main diagonal and zeros elsewhere, except in the $(N, 1)$ position. By backward substitution, we obtain the following expression for the growth rate of firm 1:

$$g_1 = \frac{1}{1 - \gamma^N} \left[ \mu \sum_{i=1}^{N-1} \gamma^i + \sum_{i=1}^{N-1} \gamma^i \epsilon_{i+1} \right]$$

Hence, the variance of a typical firm is given by

$$V(g) = \frac{\sigma_\epsilon^2 (1 - \gamma^{2N})}{(1 - \gamma^2)(1 - \gamma^N)^2}$$

$\square$

- Proof of Proposition 3

*Proof.* Next, we consider a network structure in which each firm is connected to all other firms and all links have the same weight on the other firms' growth rate. That weight is given by $\frac{1}{N-1}$. The weighting matrix $\boldsymbol{W}$ has zeros on the diagonal and $\frac{1}{N-1}$ everywhere else. The growth equation is given by $(\boldsymbol{I} - \gamma \boldsymbol{W})^{-1}(\boldsymbol{\mu} + \boldsymbol{\varepsilon})$. $(\boldsymbol{I} - \gamma \boldsymbol{W})$ has ones on the main diagonal and $\frac{-\gamma}{N-1}$ everywhere else.

$(\boldsymbol{I} - \gamma \boldsymbol{W})^{-1}$ has $\frac{(N-2)\gamma-(N-1)}{\gamma^2+(N-2)\gamma-(N-1)}$ on the diagonal and $\frac{-\gamma}{\gamma^2+(N-2)\gamma-(N-1)}$. We need to compute the diagonal elements of the variance-covariance matrix. This yields the following expression:

$$
\begin{aligned}
V\left(g_i\right) &= \sigma_\varepsilon^2 diag\left([\boldsymbol{I}-\gamma\boldsymbol{W}]^{-1}[\boldsymbol{I}-\gamma\boldsymbol{W}]^{-1}\right) \\
&= \sigma_\varepsilon^2 \left[\frac{((N-2)\gamma-(N-1))^2+(N-1)\gamma^2}{(\gamma^2+(N-2)\gamma-(N-1))^2}\right] \\
&= \sigma_\varepsilon^2 \left[\frac{\left(\frac{N-2}{N-1}\gamma-1\right)^2+\frac{1}{N-1}\gamma^2}{\left(\frac{1}{N-1}\gamma^2+\frac{N-2}{N-1}\gamma-1\right)^2}\right]
\end{aligned}
\tag{28}
$$

$\square$

- Proof of Proposition 4:

*Proof.* We use $\boldsymbol{\Lambda}$ to the diagonal eigenvalue matrix of $\boldsymbol{W}$ and $\boldsymbol{P}_t$ denotes the matrix with eigenvectors as columns. We assume the first column is the eigenvector associated with the unit eigenvalue $\boldsymbol{P}_1$. The elements of $\boldsymbol{P}_1$ can be interpreted as the total strength of all the direct and indirect business connections with $i$. Assume that the variance-covariance matrix is simply $\sigma_\epsilon \boldsymbol{I}$. We use the following eigenvector decomposition for square matrices:

$$
\boldsymbol{W}_t = \boldsymbol{P}_t \boldsymbol{\Lambda}_t \boldsymbol{P}_t^{-1}
$$

Hence, we can state the matrix inverse $(\boldsymbol{I}-\gamma\boldsymbol{W}_t)^{-1}$ as:

$$
\boldsymbol{P}_t \left[\sum_{j=0}^{\infty}\gamma^j\boldsymbol{\Lambda}_t^j\right]\boldsymbol{P}_t^{-1}.
\tag{29}
$$

The matrix inverse $\left(\boldsymbol{I}-\gamma\boldsymbol{W}_t'\right)^{-1}$ is given by:

$$
\boldsymbol{P}_t^{-1,'}\left[\sum_{j=0}^{\infty}\gamma^j\boldsymbol{\Lambda}_t^j\right]\boldsymbol{P}_t'.
$$

That delivers the following expression for the variance as a function of the eigenvalues of the $\boldsymbol{W}$ matrix:

$$
\boldsymbol{V}_t\left(\boldsymbol{g}_{t+1}\right) = \sigma_\epsilon^2\boldsymbol{P}_t\left[\sum_{j=0}^{\infty}\gamma^j\boldsymbol{\Lambda}_t^j\right]\boldsymbol{P}_t^{-1}\boldsymbol{P}_t^{-1,'}\left[\sum_{j=0}^{\infty}\gamma^j\boldsymbol{\Lambda}_t^j\right]\boldsymbol{P}_t',
$$

where $\boldsymbol{\Lambda}$ denotes the diagonal eigenvalue matrix and $\boldsymbol{P}$ denotes the matrix with eigenvectors as columns. We assume the first column is the eigenvector associated with the unit eigenvalue $\boldsymbol{P}_{t,1}$. The elements of $P_{t,1}$ can be interpreted as the long-run strength of the direct and indirect downstream business connections to $i$. The $(i,j)$-th element of expression (30) can be stated as:

$$
A_t(g)[i,j] = \frac{P_{t,i,1}P_{t,1,j}^{-1}}{(1-\gamma\lambda_1)}+\frac{P_{t,i,2}P_{t,2,j}^{-1}}{(1-\gamma\lambda_2)}+\ldots+\frac{P_{t,i,N}P_{t,N,j}^{-1}}{(1-\gamma\lambda_N)}
$$

Hence, we know that the following inequality obtains:

$$
A_t(g)[i,j] \geq \left(\frac{P_{t,i,1}P_{t,1,j}^{-1}}{(1-\gamma\lambda_1)}\right)
$$

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The variance of $g_i$ is given by $\sum_k \left(A_t(g)[i,k]\right)^2$. Hence, the lower bound is given by:

$$V_t(g)[i,i] \geq \sigma_\epsilon^2 \sum_k \left(\frac{P_{t,i,1}P_{t,1,k}^{-1}}{(1-\gamma\lambda_1)}\right)^2$$

The result follows directly from the above expression for the variances. We only consider the terms associated with the largest eigenvalue of unity to derive a lower bound. $\square$

- Proof of Proposition 5:

*Proof.* Note $E[b_{i,j,t}] = p_{i,t}$, and $b_{i,j,t}$ is independent of customer $j$'s size. By the law of large numbers, this implies that

$$\frac{1}{N}\sum_i b_{i,j,t}S_{j,t}^x \to E[b_{i,j,t}S_{j,t}^x] = p_{i,t}E[S_{j,t}^x], \ x = 1,2.$$

Therefore, for sufficiently large $N$, a supplier's customer concentration Herfindahl is approximately

$$H_{i,t} = \frac{\frac{1}{N}\sum_j b_{i,j,t}S_{j,t}^2}{N\left(\frac{1}{N}\sum_j b_{i,j,t}S_{j,t}\right)^2} \approx \frac{E[S_{j,t}^2]}{p_{i,t}NE[S_{j,t}]^2}.$$

This bears close resemblance to the Herfindahl of the economy-wide size distribution, $H_t$. By the same law of large numbers rationale,

$$H_t = \frac{1}{N}\sum_i \frac{S_{i,t-1}^2}{\left(\sum_i S_{i,t-1}\right)^2} \approx \frac{E[S_{j,t-1}^2]}{NE[S_{j,t-1}]^2}.$$

Together, these results imply the following common factor structure for growth rate volatilities of firms in this network economy:

$$V_t\left(g_{i,t+1}\right) \approx \sigma_\varepsilon^2\left(1 + \frac{1}{p_{i,t}}\gamma^2 H_t\right). \tag{30}$$

$\square$

- Proof of Corollary 2

*Proof.* If the firm size distribution is log normal at $t$, then

$$E[S_{i,t}^x] = \exp\left(x\mu_{s,t} + \frac{x^2}{2}\sigma_{s,t}^2\right)$$

where $\mu_{s,t} \equiv E[\log S_{i,t}]$ and $\sigma_{s,t}^2 \equiv V(\log S_{i,t})$. As a result, the firm size Herfindahl is

$$H_t \approx \frac{E[S_{j,t}^2]}{NE[S_{j,t}]^2} = \frac{1}{N}\exp\left(\sigma_{s,t}^2\right).$$

Hence, in the first-order approximation, the common variance dynamics for all firms are determined by cross section standard deviation of the size distribution, $\sigma_{s,t}^2$:

$$V(g_{i,t+1}) \approx \sigma_\varepsilon^2\left(1 + \gamma^2\frac{\exp\left(\sigma_{s,t}^2\right)}{Np_{i,t}}\right).$$

$\square$

- Proof of Corollary 3

    *Proof.* If the firm size distribution is not log normal at $t$, then we can still use the cumulant-generating function to characterize volatility. Note that the expected value of any moment of firm size is given by:

    $$E[S_t^x] = \exp\left(\sum_{j=1}^{\infty} \frac{x^j}{j!} \kappa_{s,j,t}\right)$$

    where $\mu_{s,j,t} \equiv E[(\log S_t)^j]$ denotes the $j$-th central conditional moment and $\kappa_{1,s,t} = \mu_{1,s,t}$, $\kappa_{2,s,t} = \mu_{2,s,t}$, $\kappa_{3,s,t} = \mu_{3,s,t}$ and $\kappa_{4,s,t} = \mu_{4,s,t} - 3\mu_{2,s,t}^2$. As a result, the economy-wide Herfindahl is given by:

    $$H_t \approx \frac{E[S_{j,t}^2]}{NE[S_{j,t}]^2} = \frac{1}{N} \exp\left(\sum_{j=2}^{\infty} \frac{2^j - 2}{j!} \kappa_{s,j,t}\right).$$

    Hence, in the first-order approximation, the common variance dynamics for all firms are determined by cross section moments of the size distribution:

    $$V(g_{i,t+1}) \approx \sigma_\varepsilon^2 \left(1 + \frac{\gamma^2}{Np_{i,t}} \exp\left(\sum_{j=2}^{\infty} \frac{2^j - 2}{j!} \kappa_{s,j,t}\right)\right).$$

    $\square$

- Proof of Proposition 8

    *Proof.* The first order approximation for $g_{a,t+1}$ is

    $$g_{a,t+1} \approx \sum_j \varepsilon_{j,t+1} \left(\tilde{S}_{j,t} + \gamma \sum_i \tilde{S}_{i,t} w_{i,j,t}\right).$$

    We can rewrite this expression (suppressing $t$) as

    $$\sum_i \tilde{S}_i w_{i,j} = \frac{\sum_i S_i b_{i,j} S_j}{\sum_i S_i \sum_j b_{i,j} S_j} \approx \frac{S_j \sum_i S_i \frac{b_{i,j}}{p_i}}{N^2 E[S_i]^2}.$$

    Under weak regularity, $\frac{1}{N} \sum_i S_i \frac{b_{i,j}}{p_i} \xrightarrow{N \to \infty} E\left[S_i \frac{b_{i,j}}{p_i}\right] = E\left[S_i E\left[\frac{b_{i,j}}{p_i}|S_i\right]\right] = E[S_i]$, where the last equality follows from $b_{i,j}$ being Bernoulli$(p_i)$. As a result, $\sum_k \tilde{S}_k w_{k,j} \approx S_j/(NE[S_j]) = \tilde{S}_j$.

    Noting that $\sum_k \tilde{S}_{k,t}^2 = H_t$ and $\sum_k \tilde{S}_{k,t} w_{k,j,t} \approx S_{j,t}/(NE[S_{j,t}])$, the aggregate growth rate variance can be approximated as

    $$\begin{aligned} V(g_{a,t+1}) &\approx \sigma_\varepsilon^2 \sum_j \left(\tilde{S}_{j,t} + \gamma \sum_i \tilde{S}_{i,t} w_{i,j,t}\right)^2 \\ &\approx \sigma_\varepsilon^2 H_t (1+\gamma)^2. \end{aligned} \tag{31}$$

    $\square$

- Proof of Proposition 9:

*Proof.* We will derive a lower bound on the aggregate variance. We use the following notation to denote the matrix inverse:

$$\boldsymbol{v} = (\boldsymbol{I} - \gamma \boldsymbol{W}_t)^{-1} \quad = \quad \boldsymbol{I} + \gamma \boldsymbol{W}_t + \gamma^2 \boldsymbol{W}_t^2 + \gamma^3 \boldsymbol{W}_t^3 +$$

Because the $\boldsymbol{W}$ matrix only has positive entries, it follows that

$$\tilde{\boldsymbol{S}}_t' \boldsymbol{v} \geq \tilde{\boldsymbol{S}}_t'(\boldsymbol{I} + \gamma \boldsymbol{W}_t)$$

This in turn implies that the following inequality holds:

$$\tilde{\boldsymbol{S}}_t' \boldsymbol{v} \geq \tilde{\boldsymbol{S}}_t'(\boldsymbol{I} + \gamma \boldsymbol{W}_t)$$

As a result, the following inequality holds for the quadratic expression:

$$\tilde{\boldsymbol{S}}_t' \boldsymbol{v} \boldsymbol{v}' \tilde{\boldsymbol{S}}_t' \geq \tilde{\boldsymbol{S}}_t'(\boldsymbol{I} + \gamma \boldsymbol{W}_t)(\boldsymbol{I} + \gamma \boldsymbol{W}_t') \tilde{\boldsymbol{S}}_t$$

The right hand side of this inequality is given by:

$$\tilde{\boldsymbol{S}}_t' \tilde{\boldsymbol{S}}_t + 2 \tilde{\boldsymbol{S}}_t' \gamma \boldsymbol{W}_t \tilde{\boldsymbol{S}}_t + \tilde{\boldsymbol{S}}_t' \gamma^2 \boldsymbol{W}_t \boldsymbol{W}_t' \tilde{\boldsymbol{S}}_t$$

The aggregate variance is given by $\sigma_\epsilon^2 \tilde{\boldsymbol{S}}_t' \boldsymbol{v} \boldsymbol{v}' \tilde{\boldsymbol{S}}_t'$. Hence, we have derived the following lower bound on the aggregate variance:

$$\tilde{\boldsymbol{S}}_t' \boldsymbol{v} \boldsymbol{v}' \tilde{\boldsymbol{S}}_t' \geq H_t + 2\gamma \sum_{i=1}^N \sum_{j=1}^N w_{ij,t} \tilde{S}_{i,t} \tilde{S}_{j,t} + \gamma^2 \sum_{i=1}^N \sum_{k=1}^N \tilde{S}_{i,t} \tilde{S}_{k,t} \sum_{j=1}^N w_{ij,t} w_{kj,t}$$

Using the definition of the weights, we obtain the following expression:

$$H_t + 2\gamma \frac{\sum_{i=1}^N \sum_{j=1}^N b_{ij,t} \tilde{S}_{i,t} \tilde{S}_{j,t}^2}{\sum_i b_{ij} \tilde{S}_j} + \gamma^2 \frac{\sum_{i=1}^N \tilde{S}_{i,t} \sum_{k=1}^N \tilde{S}_{k,t} \sum_{j=1} b_{ij,t} b_{kj,t} \tilde{S}_{j,t}^2}{\sum_i b_{ij} \tilde{S}_j \sum_k b_{jk} \tilde{S}_k}$$

This can be further simplified to obtain the following expression:

$$H_t + 2\gamma \frac{\sum_{i=1}^N S_{i,t} \sum_{j=1}^N b_{ij,t} S_{j,t}^2}{\sum_i b_{ij} S_j (\sum_l S_l)^2} + \gamma^2 \frac{\sum_{k=1}^N S_{k,t} \sum_{i=1}^N S_{i,t} \sum_{j=1} b_{ij,t} b_{kj,t} S_{j,t}^2}{\sum_i b_{ij} S_j \sum_k b_{jk} S_k (\sum_l S_l)^2}$$

Note $E[b_{i,j,t}] = p_{i,t}$, and $b_{i,j,t}$ is independent of customer $j$'s size. By the law of large numbers, this implies that

$$\frac{1}{N} \sum_j b_{i,j,t} b_{k,j,t} S_{j,t}^x \to E[b_{i,j,t} b_{k,j,t} S_{j,t}^x] = p_{i,t} p_{k,t} E[S_{j,t}^x], \ x = 1, 2.$$

We also know that:

$$\frac{1}{N} \sum_j \frac{b_{i,j,t}}{p_{i,t}} S_{j,t}^x \to E[S_{j,t}^x], \ x = 1, 2.$$

Therefore, for sufficiently large $N$, the aggregate variance is bounded below by:

$$Var(g_t^a) \geq \sigma_\epsilon^2 \frac{E[S_i^2]}{N E[S_i]^2} \left(1 + 2\gamma + \gamma^2\right)$$

for large $N$. This expression can be succinctly stated as follows:

$$Var(g_t^a) \geq H_t(1 + \gamma)^2$$

□

- Proof of Proposition 7

*Proof.* The covariance between an individual growth rate and aggregate growth is approximated by

$$
Cov_t(g_{i,t+1}, g_{a,t+1}) \approx Cov_t\left(\varepsilon_{i,t+1} + \gamma \sum_j w_{i,j,t}\varepsilon_{j,t+1} \,,\, \sum_j \varepsilon_{j,t+1}\left[\tilde{S}_{j,t} + \gamma \sum_i \tilde{S}_{i,t} w_{i,j,t}\right]\right)
$$

$$
\approx \sigma_\varepsilon^2(1+\gamma)\left(\frac{S_{i,t}}{NE[S_{i,t}]} + \gamma H_t\right)
$$

Therefore the regression coefficient is approximated by

$$
\beta_{i,t} = \frac{Cov_t(g_{i,t+1}, g_{a,t+1})}{V(g_{a,t+1})} \approx \frac{1}{1+\gamma}\left(\gamma + S_{i,t}\frac{E[S_{i,t}]}{E[S_{i,t}^2]}\right).
$$

From here, the close similarity in volatility factor structure for raw growth rate volatility and residual volatility becomes apparent. Residual growth from a factor model, defined as $g_{i,t+1}^{res} = g_{i,t+1} - \beta_i g_{a,t+1}$ has variance

$$
\begin{aligned}
V_t(g_{i,t+1}^{res}) &= V_t(g_{i,t+1}) - 2\beta_{i,t} Cov_t(g_{i,t+1}, g_{a,t+1}) + \beta_{i,t}^2 V_t(g_{a,t+1}) \\
&= V_t(g_{i,t+1}) - Cov_t(g_{i,t+1}, g_{a,t+1})^2/V_t(g_{a,t+1}) \\
&\approx \sigma_\varepsilon^2\left(1 + \frac{1}{p_{i,t}}\gamma^2 H_t - \left[\frac{S_{i,t}}{NE[S_{i,t}]} + \gamma H_t\right]^2\right).
\end{aligned}
$$

□

- Proof of Proposition 6:

*Proof.* If $p_{i,t} = S_{i,t}/\left(Z\sum_j(S_{j,t})\right)$, then approximate log normality of $S_{i,t}$ implies $V(g_{i,t+1})$ is also approximately log normal. Furthermore, the parameters of the $V(g_{i,t+1})$ distribution depend only on the dispersion in firm sizes. This is evident from Equation 19, which implies

$$
\begin{aligned}
\log\left(V(g_{i,t+1}) - \sigma_\varepsilon^2\right) &= \log(\sigma_\varepsilon^2\gamma^2 Z) - \log(S_{i,t}) + \log\left(\sum_j S_{j,t}\right) + \log(H_t) \\
&= \log(\sigma_\varepsilon^2\gamma^2 Z) - \log(S_{i,t}) + \log(NE[S_{j,t}]) + \log(H_t).
\end{aligned}
$$

Assuming that $S_{i,t}$ is log normal and noting that $\log\left(\sum_j S_{j,t}\right) \approx \log(NE[S_{j,t}])$, we see that $V(g_{i,t+1})$ is a shifted log normal where

$$
E\left[\log\left(V(g_{i,t+1}) - \sigma_\varepsilon^2\right)\right] \approx \log(\sigma_\varepsilon^2\gamma^2 Z) + \frac{3}{2}\sigma_{S,t}^2
$$

and

$$
V\left(\log\left(V(g_{i,t+1}) - \sigma_\varepsilon^2\right)\right) \approx \sigma_{S,t}^2.
$$

□

- Proof of Proposition 10

*Proof.* For large $N$, we can approximate the first-order growth dynamics of firm $i$ as:

$$g_{i,t+1} \approx \mu + \eta_{t+1}, \eta_{t+1} \sim N\left(0, \sigma_\epsilon^2\left(1 + \frac{\gamma^2}{\widetilde{S}_{i,t}N}Z\exp\left(\sum_{j=2}^\infty \frac{2^j-2}{j!}\kappa_{s,j,t-1}\right)\right)\right)$$

This follows directly from corollary 3 and the specification for $p_i$. For large $N$, this can be further simplified to yield:

$$g_{i,t+1} \approx \mu + \eta_{t+1}, \eta_{t+1} \sim N\left(0, \sigma_\epsilon^2\left(1 + \frac{\gamma^2}{S_{i,t}}Z\exp\left(\sum_{j=2}^\infty \frac{2^j-1}{j!}\kappa_{s,j,t-1}\right)\right)\right)$$

The log size of firm $i$ at time $t$ is given by $\log S_{i,t} = \log S_{i,0} + \sum_{\tau=1}^t g_{i,\tau}$. We use $\sigma_{s_i,t}^2$ to denote the variance of $\log S_{i,t}$ conditional on $S_{i,0}$. Hence, the variance of the log size distribution at $t$ in the first-order version of this economy is given by:

$$\sigma_{s_i,t}^2 \approx \sigma_{s_i,t-1}^2 + \sigma_\epsilon^2\left(1 + \frac{\gamma^2}{\widetilde{S}_{i,t-1}N}Z\,\exp\left(\sum_{j=2}^\infty \frac{2^j-2}{j!}\kappa_{s,j,t-1}\right)\right), t \geq 1,$$

which is the variance of the cumulative growth rate of firm $i$. This implies that the average variance of the size of firms in the first-order version of this economy is given by:

$$E[\sigma_{s_i,t}^2] \approx E[\sigma_{s_i,t-1}^2] + \sigma_\epsilon^2\left(1 + \gamma^2 Z\exp\left(\sum_{j=2}^\infty \frac{2^j-1+(-1)^j}{j!}\kappa_{s,j,\tau-1}\right)\right), t \geq 1$$

Finally, note that:
$$Var(\log S_t) \geq E[\sigma_{s_i,t}^2] = E[Var(\log S_t|S_0)].$$

This follows directly from the Cauchy-Schwarz inequality.

Hence, in the first-order version of our economy, for large $N$, there is a lower bound on the cross-sectional variance of the log size distribution given by:

$$E[\sigma_{s_i,0}^2] + \sum_{\tau=1}^t \sigma_\epsilon^2\left(1 + \gamma^2 Z\exp\left(\sum_{j=2}^\infty \frac{2^j-1+(-1)^j}{j!}\kappa_{s,j,\tau-1}\right)\right), t \geq 1.$$

In the log-normal case, the last sum reduces to $\exp(2\sigma_{s,\tau-1}^2)$:

$$E[\sigma_{s_i,0}^2] + \sum_{\tau=1}^t \sigma_\epsilon^2\left(1 + \gamma^2 Z\exp(2\sigma_{s,\tau-1}^2)\right), t \geq 1$$

The result stated in the proposition follows directly if the higher-order cumulants are positive.

$\square$

# A Downstream Transmission

Acemoglu et al. (2012) derive a static version of (2) as the equilibrium outcome in a multi-sector production economy in a constant-returns to scale economy who has Cobb-Douglas preferences defined over all of the $N$ different commodities produced in this economy. Theirs is also a directed network, but in their version the productivity shocks are transferred downstream from suppliers to customers. Consider the growth equation for (customer) firm $j$:

$$g_{j,t+1} = \mu_g + \gamma \sum_{k=1}^{N} w_{k,j,t}\, g_{k,t+1} + \varepsilon_{j,t+1}. \tag{32}$$

In this economy, the weights $w_{kj}$ are the input weights of good $k$ for firm $j$ in the production of commodity $j$. $\gamma$ measures the input share in GDP. In matrix notation, this amounts to:

$$\boldsymbol{g}_{t+1} = \boldsymbol{\mu}_g + \gamma \boldsymbol{W}_t' \boldsymbol{g}_{t+1} + \boldsymbol{\varepsilon}_{t+1}. \tag{33}$$

This specification emphasizes upstream supply shocks as drivers of firm-level volatility, while our model focuses on downstream demand shocks as the only drivers.

# B Retailers

In the data, we also consider retailers who do not have any out-degrees in the standard sense. Our simple model can be extended to include retailers. Consider a group of retailers $l = N+1, .., N+k$. In the strictest interpretation of the model, retailers have $w_{lm} = 0, m = 1, \ldots, N$. They contribute $k$ zero rows to the adjacency matrix $\boldsymbol{W}$. However, if markets are incomplete, then some of the labor income risk that is specific to firms would show up in the consumption decisions of workers at these firms. That in turn would expose retail firms to some of the upstream risk.

Let $v_{l,m,t}$ denote the link strength of retailer $l$ to customers working at supplier $m$.

$$g_{i,t+1} = \mu_g + \gamma \sum_{j=1}^{N+k} w_{i,j,t}\, g_{j,t+1} + \varepsilon_{i,t+1}, i = 1, \ldots, N. \tag{34}$$

$$g_{l,t+1} = \mu_g + \psi \sum_{m=1}^{N+k} v_{l,m,t}\, g_{m,t+1} + \varepsilon_{l,t+1}, l = N+1, .., N+k. \tag{35}$$

$\psi$ governs how much firm-specific idiosyncratic risk is transferred to consumption; when markets are complete, $\psi = 0$. We adopt specification (17) for these retail-customer weights and impose that they sum to one. Larger retailers are connected to a larger number of consumers, and larger consumers are more important. Retail firms create network leakage because of the zero rows in the $\boldsymbol{W}$ matrix.

# C Approximate Log Normality of Size and Volatility

We first we present evidence that the cross section distribution of size and volatility are approximately log normal. There is an extensive literature studying the firm size distribution that we do not cover here. We merely note that our sample of firm sizes is approximately log normal. Hall (1987) characterized the literature on US firm sizes saying "The size distribution of firms conforms fairly well to the log normal, with possibly some skewness to the right." Axtell (2001) provides evidence that firm sizes follow a power law in a large sample including all private US firms. Cabral and Mata (2003) argue that firm sizes evolve toward a log normal distribution over time. Rossi-Hansberg (2007) and Wright show that establishment and enterprise

## Figure 7: Log Firm Size: Empirical Density Versus Normal Density

### Panel A: Size Measured as Market Equity



### Panel B: Size Measured as Sales
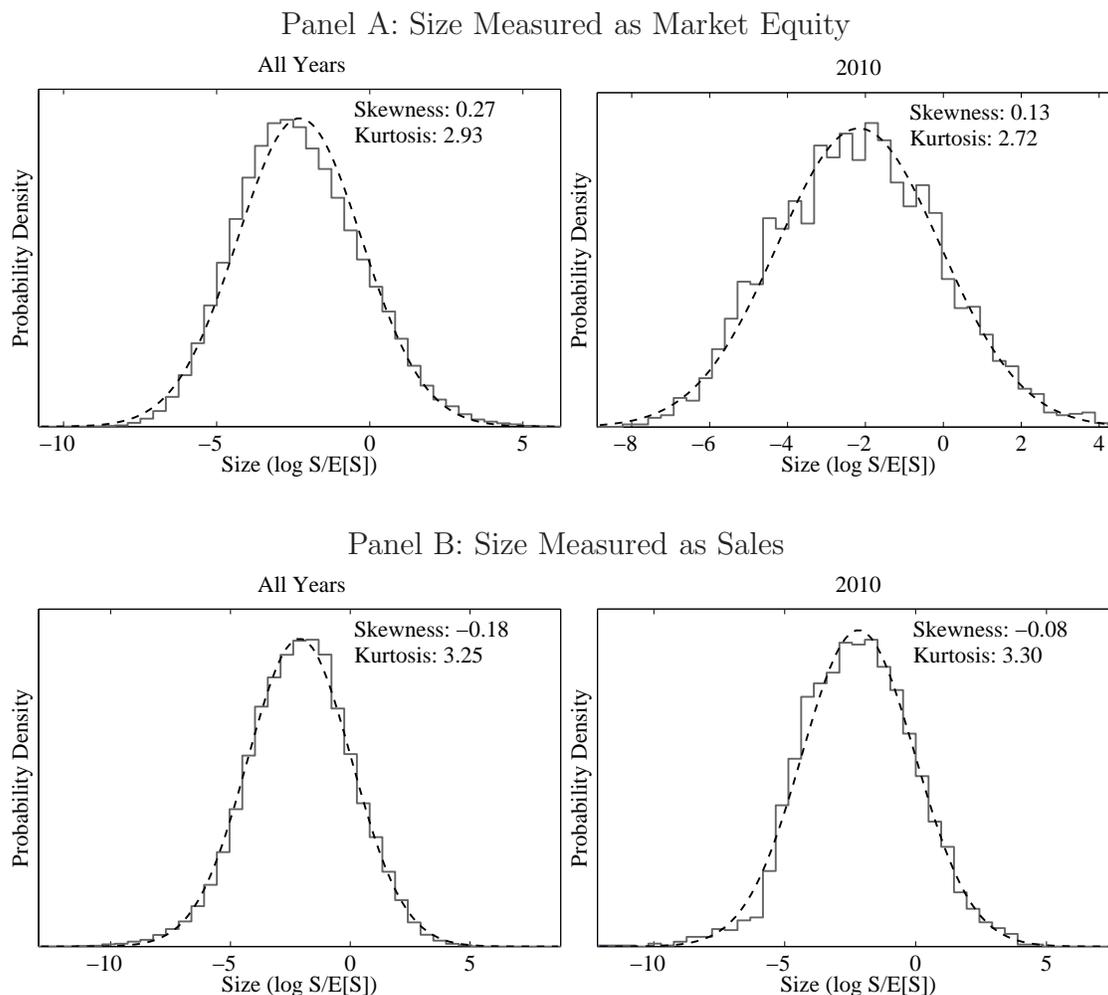


*Notes:* The figure plots histograms of the empirical cross section distribution of log annual firm market equity (Panel A) and log annual firm sales (Panel B). The left-hand histogram pools all years (1926-2010) and the right-hand histogram is the one-year snapshot for 2010. Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

sizes for public and private US non-farm businesses have tails that are thinner than power law. We discuss the association between public and private firm size distributions in Appendix F.

Figure 7 plots histograms of the empirical cross section distribution of firms' market equity value (Panel A) and sales (Panel B), taking the log of both quantities. The left figure in each panel shows the distribution of firm sizes pooling all firm-years from (1926–2010 for market equity, 1970–2010 for sales) and the right figure shows a one-year snapshot for 2010. Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure also reports the skewness and kurtosis of the data in the histogram.

The pooled empirical distribution, and the distributions for all individual years, appear nearly normally distributed. They demonstrate only slight skewness (always less than 0.5 in absolute value) and do not

Figure 8: LOG VOLATILITY: EMPIRICAL DENSITY VERSUS NORMAL DENSITY

Panel A: Volatility Measured from Market Returns



Panel B: Volatility Measured from Sales Growth
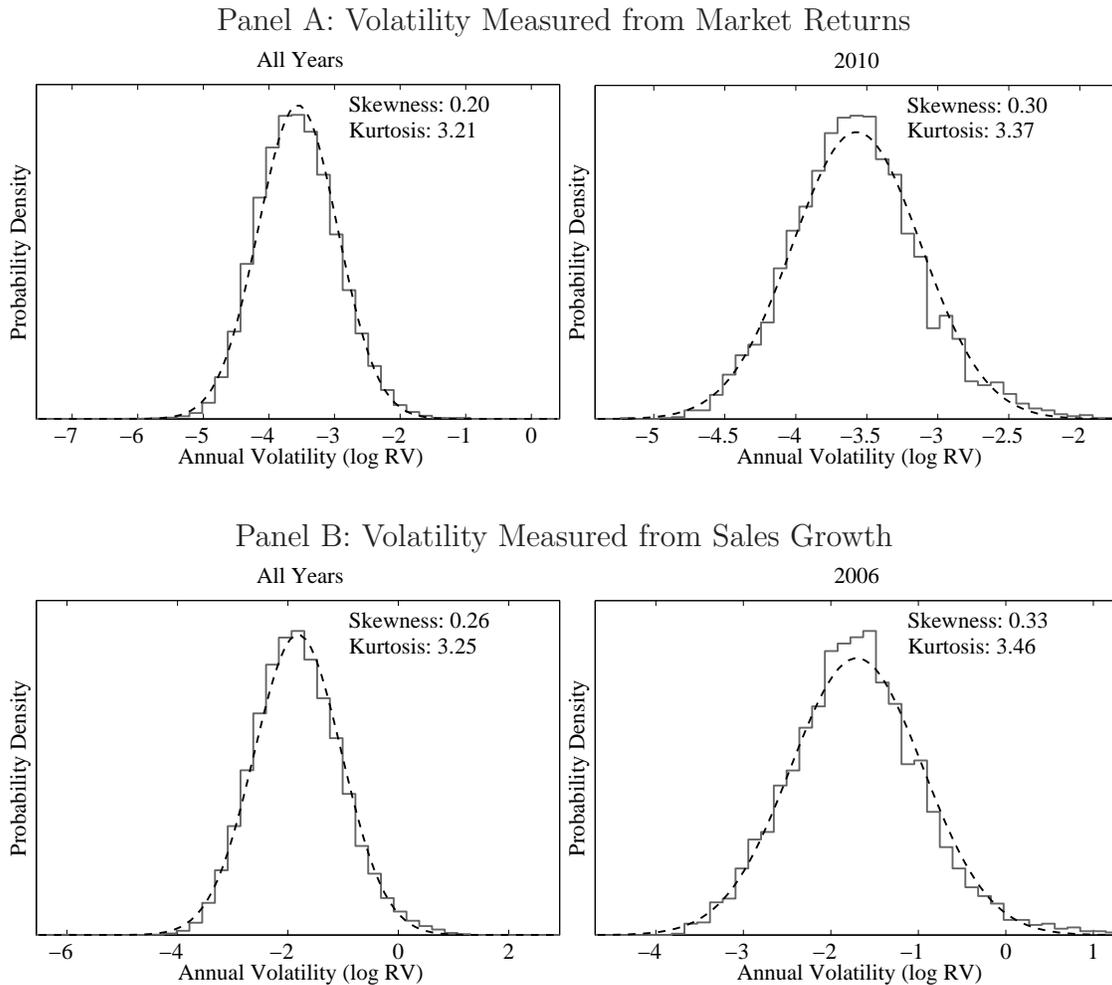


*Notes:* The figure plots histograms of the empirical cross section distribution of annual firm-level volatility (in logs). Panel A reports return volatility where, within each calendar year, we calculate the standard deviation of daily returns for each stock. Panel B reports sales growth volatility where, for each year $t$, we calculate volatility as the standard deviation of quarterly observations of year-on-year sales growth for each stock in calendar years $t$ to $t + 4$. The left-hand histogram pools all years (1926-2010) and the right-hand histogram is the one-year snapshot for 2010 for returns and 2006 for sales growth (since sales growth volatility is calculated over a five-year window).

possess substantive leptokurtosis (always less than 3.3).

Figure 8 shows cross section distributions of yearly return volatility (Panel A) and sales growth volatility (Panel B) in logs for all CRSP/Compustat firms. Volatility also appears to closely fit a log normal distribution, with skewness no larger than 0.4 and kurtosis never exceeding 3.7. Figure 8 demonstrates near log normality of total return and growth rate volatility. Kelly, Lustig and Van Nieuwerburgh (2012) show that the same feature holds for idiosyncratic volatility.

# D   Upstream vs Downstream

We first investigate our modeling assumption that shocks are transmitted from customers to suppliers, and not the reverse. Table 10 reports estimates for how firms' sales growth rates are affected by growth of firms in their customer-supplier networks. In column (1) we estimate a pooled OLS regression model for upstream growth rate transmission that takes the form $g_{i,t+1} = a + b\sum_j w_{i,j,t}g_{j,t+1} + e_{i,t+1}$ where the dependent variable is the sales growth rate of supplier $i$ in year $t+1$ and $\sum_j w_{i,j,t}g_{j,t+1}$ is the network effect of $i$'s customers on $i$'s growth rate. The weights are taken directly from the from Compustat linkage data. The regression coefficient $b$, which is the spatial equivalent of an autoregression coefficient, is analogous to $\gamma$ in our model. We find that a customer's weighted growth rate has an influence of $\hat{b} = 0.90$ on its immediate suppliers, $0.90^2$ on its suppliers suppliers, and so forth through the network. Thus a customer's growth rate shock has a half-life of 8 steps through the network.

Column (2) runs the same regression but also controls for $g_{\mathrm{agg},t+1}$, the sales-weighted average growth rate for all firms in the sample. This regression allows us to test whether customer linkages have explanatory for a supplier's growth rate after controlling for a simple one-factor structure in growth rates where all firms have the same loading on the factor. After controlling for aggregate growth, the coefficient on weighted customer growth rates is 0.79 and is highly statistically significant (in all cases standard errors are clustered by firm and year).

In column (3) we allow for supplier specific loadings on the aggregate growth factor $g_{\mathrm{agg},t+1}$ by first running univariate time series regressions of the form $g_{i,t+1} = c_i + d_i g_{\mathrm{agg},t+1} + e_{i,t+1}$, then running a second-stage pooled OLS regression of the residuals on the network-based growth effect: $e_{i,t+1} = a + b\sum_j w_{i,j,t}g_{j,t+1} + v_{i,t+1}$. We find a second stage coefficient on the upstream network effect of 0.64, which is again highly significant. Columns (4), (5) and (6) report downstream transmission model estimates, which take the same form as (1), (2) and (3) except the dependent variable is the growth rate of each customer, and the network effect is calculated as the weighted average growth rate of suppliers a given customer (with weights being the value of customer purchases from each supplier divided by the total sales of the customer firm). The estimated downstream transmission effect is economically small, only $-0.02$. While this estimate is marginally significant, the sign is negative and we view these results as a failure to reject the null of no downstream effect.

# E   Network Overview

Figure 9 provides a summary of network connections for customers and suppliers in the Compustat linkage data. The left panel shows the distribution of number of links by supplier (out-degree) on a log-log scale, while the right panel shows the distribution of links by customer (in-degree). Out-degrees range between one and 24, while in-degrees range from one to 130. We expect that both of these distributions are highly distorted due to the truncated nature of the data. Out-degrees can reach 24 since some suppliers (23%) voluntarily report customers that fall below the 10% sales threshold. The maximum out-degrees falls to 5 when we strictly impose the 10% sales truncation. Figure 10 reports histograms of weights of customer-supplier sales linkages pooling all supplier-year observations. The distribution for the raw data, in which some suppliers voluntarily report customers below the 10% sales threshold, is on the left. The right panel shows the weight distribution when we strictly impose the 10% sales truncation in our calibration below.

These correlations are visualized in Figure 11, which presents a snapshot of the Compustat customer-supplier network in 1980, zooming in on a region of the network dominated by retail firms (Sears, KMart and JC Penney), automotive manufacturers (Ford and General Motors), technology/telecom firms (IBM and AT&T) and industrials (General Electric and United Technologies). The size of each nodes is proportional to a firm's log sales, and edges between nodes are proportional to $w_{i,j,t}$. Arrows on edges point from customer to supplier.

## Figure 9: Customer-Supplier Network Degree Distributions
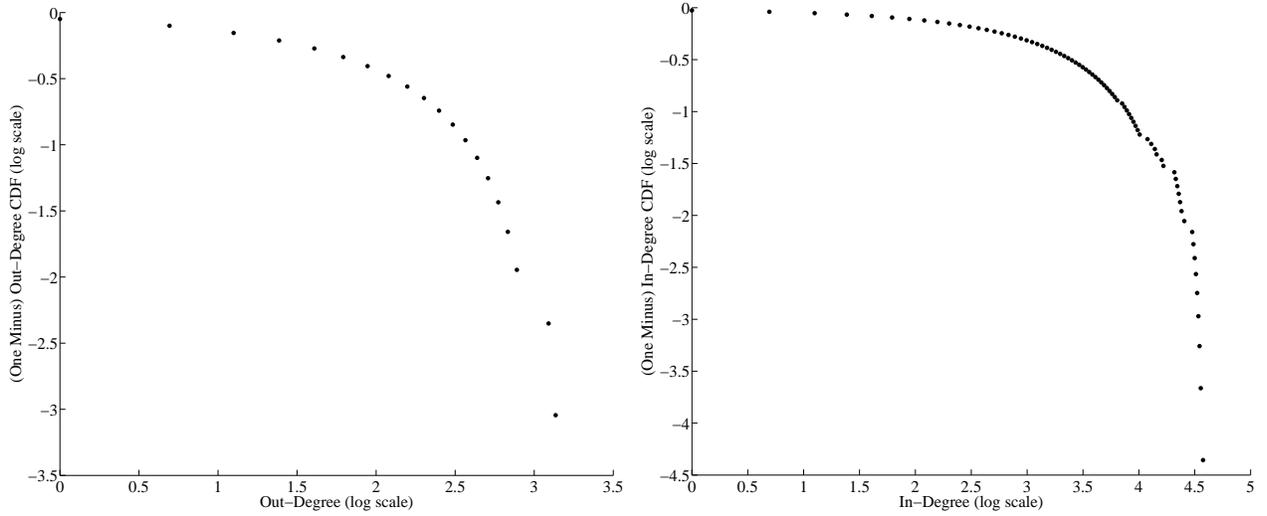


*Notes:* The figure plots log-log survivor plots of the out-degree and in-degree distributions of the Compustat customer-supplier network data pooling all firm-years for 1980–2009.

## Figure 10: Customer-Supplier Network Linkage Weights
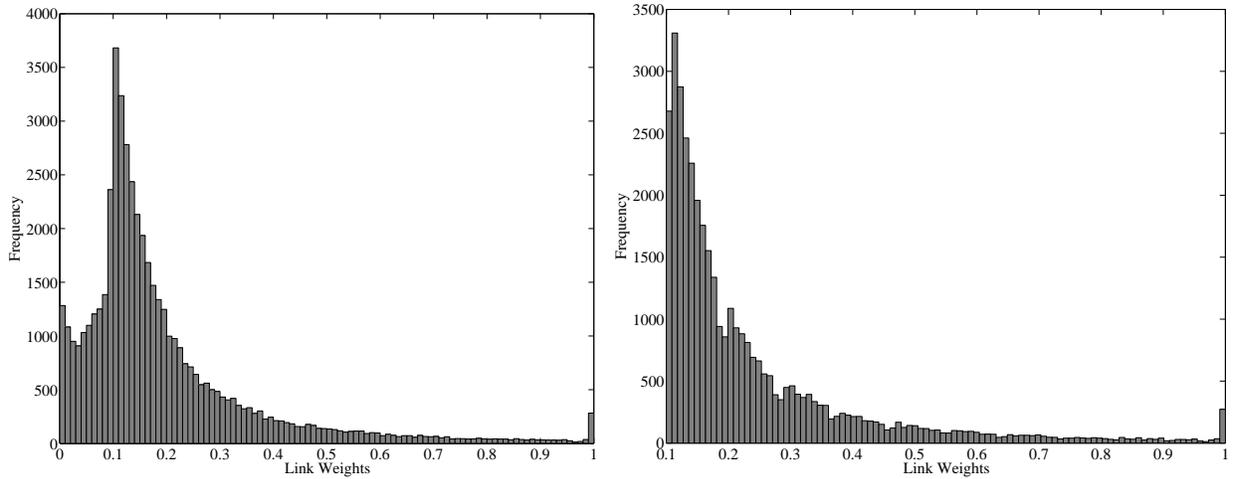


*Notes:* The figure plots the histogram of Compustat customer-supplier sales weights for the raw data (left panel), and when we strictly imposed the 10% sales threshold in our calibration. Plots pool all firm-years for 1980–2009.

5

Table 10: Upstream versus Downstream Growth Rate Transmission

| | Supplier Growth | | | Customer Growth | | |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) |
| Linkages Growth | 0.901 | 0.788 | 0.642 | -0.022 | -0.024 | -0.022 |
| | *8.63* | *7.91* | *5.17* | *-2.24* | *-2.45* | *-2.06* |
| Aggregate Growth | | 0.760 | | | 1.045 | |
| | | *6.74* | | | *8.72* | |
| $R^2$ (%) | 3.11 | 4.50 | 1.84 | 0.13 | 3.45 | 0.13 |

*Notes:* The table reports estimates for how firms' sales growth rates are affected by growth of firms in their customer-supplier networks. In column (1) we estimate a pooled OLS regression model for upstream growth rate transmission that takes the form $g_{i,t+1} = a + b\sum_j w_{i,j,t}g_{j,t+1} + e_{i,t+1}$ where the dependent variable is the sales growth rate of supplier $i$ in year $t+1$ and $\sum_j w_{i,j,t}g_{j,t+1}$ is the network effect of $i$'s customers on $i$'s growth rate (calculated from Compustat linkage data). Column (2) runs the same regression but also controls for $g_{\text{agg},t+1}$, the sales-weighted average growth rate for all firms in the sample. In column (3) we allow for supplier specific loadings on the aggregate growth factor $g_{\text{agg},t+1}$ by first running univariate time series regressions of the form $g_{i,t+1} = c_i + d_i g_{\text{agg},t+1} + e_{i,t+1}$, then running a second-stage pooled OLS regression of the residuals on the network-based growth effect: $e_{i,t+1} = a + b\sum_j w_{i,j,t}g_{j,t+1} + v_{i,t+1}$ (only second stage estimates are reported). Columns (4), (5) and (6) report downstream transmission model estimates, which take the same form as (1), (2) and (3) except the dependent variable is the growth rate of each customer, and the network effect is calculated as the weighted average growth rate of suppliers a given customer (with weights being the value of customer purchases from each supplier divided by the total sales of the customer firm). All growth rates are de-meaned at the firm level, and standard errors are clustered by firm and year.

# F    Inclusion of Private Firms

We use Census data private sector firms sorted by employment. The data is provided by the U.S. Census Bureau, Business Dynamic Statistics. The sample covers 1977-2009. The data is annual. The census reports data in 12 employment bins ranging from $1-4$ at the low end to $10,000+$ at the high end. We construct the same bins using the Compustat employment data.

We also created a spliced series that divides the 10,000+ binds into 25 sub-bins using an imputation from the Compustat size data. At the start of the sample, Census reports 728 firms with more than 10,000 employees, while Compustat reports 677. So, we have fairly comprehensive coverage at the start of the sample. However, at the end of the sample, there are 1,975 with 10,000+ firms,only 1,016 of which show up in Compustat. There are more large, private firms at the end of the sample.

Figure 12 plots the cross-sectional variance of log employment in Census and Compustat data. The evolution of the size distribution when considering the entire universe of firms seems similar to the one in Compustat. The correlation between the cross-sectional variance of log size in Compustat and the spliced series (shown in Panel A) is 0.62. The correlation between the non-spliced Census measure (shown in Panel B) and the Compustat measure is 0.65.[20] Hence, the secular changes in the size distribution that we have documented are not specific to publicly traded firms.

---

[20]Note that the variance of log employment in the Compustat data is much larger than the variance in the Census data.

Finally, figure 13 plots the entry and exit rates for U.S. firms. The figure documents a secular decline in entry and exit rates over the sample. These secular changes may have contributed to the changes in the size distribution that we have documented.

# G    Tail Probabilities

As $N \to \infty$, the pdf converges to $f(g; G_t(S))$, with dependence on the entire distribution of firm sizes.

The law of motion for the tail distribution $G_t(S) = Prob(x > S)$ can be stated as:

$$G_{t+1}(S) = \int G_t\left(\frac{S}{\exp(g)}\right) f_t(g) d\tilde{g}, \tag{36}$$

where $f$ denotes the p.d.f. of the growth rates. Assume that the cross-sectional size distribution is log-normal. After substituting the approximate first-order approximation of growth dynamics, we obtain:

$$G_{t+1}(S) = \int G_t\left(\frac{S}{\exp(g)}\right) \frac{1}{\sigma_\epsilon \sqrt{1 + \frac{\gamma^2}{Np(S)}\exp(\sigma_{s,t-1}^2)}} \exp\left(-\frac{g}{2\sigma_\epsilon^2\left(1 + \frac{\gamma^2}{Np(S)}\exp(\sigma_{s,t-1}^2)\right)}\right) dg$$

**Proposition 11.** *In the first-order version of this economy, there exists a steady state distribution which satisfies a power law with coefficient:*

$$\zeta(S) = \frac{1}{1 + \frac{\gamma^2 ZE[S]}{S}\left(\sum_{j=2}^{\infty} \frac{2^{j-2}}{j!}\kappa_{s,j,t}\right)},$$

*provided that $\mu_g = -(1/2)\sigma_\epsilon^2$.*

*Proof.* We consider the case of a log-normal size distribution. The extension to the general case is straightforward. Suppose we conjecture the following tail distribution $G(S) = a/S^{\zeta(S)}$ and solve for a steady-state tail distribution.

$$G(S) = \int \left(\frac{\exp(g)}{S}\right)^{\zeta(S)} a \frac{1}{\sigma_\epsilon \sqrt{1 + \frac{\gamma^2}{Np(S)}\exp(\sigma_s^2)}} \exp\left(-\frac{g}{2\sigma_\epsilon^2\left(1 + \frac{\gamma^2}{Np(S)}\exp(\sigma_s^2)\right)}\right) dg$$

This can also be stated as:

$$G(S) = \frac{a}{S^{\zeta(S)}} E[\exp(g\zeta(S))|S]$$

To solve for a steady-state distribution, we have to find a function $\zeta(S)$ such that: $E[\exp(g\zeta(S))|S] = 1$ for all $S$. We need to solve for the following equation for $\zeta(S)$:

$$\log E[\exp(g\zeta(S))|S] = \mu_g\zeta(S) + (1/2)\zeta(S)^2\sigma_\epsilon^2\left(1 + \frac{\gamma^2}{Np(S)}\exp(\sigma_s^2)\right) = 0.$$

This is a quadratic equation that can be solved for $\zeta$. We assume that $\mu_g = -(1/2)\sigma_\epsilon^2$. We substitute for $p(s)$. As $N \to \infty$, $\sum_i S_i \to NE[S]$. After some simplifications, we obtain the following non-zero root of the quadratic equation:

$$\zeta(S) = \frac{1}{1 + \frac{\gamma^2 ZE[S]}{S}\exp(\sigma_s^2)}.$$

In the general case with non-normal size distribution, we obtain the same expression with the cumulant function inside the exponent. $\qquad \blacksquare$

As $S/E(S) \to \infty$, $\zeta(S) \to 1$. In this limiting case, we end up back in the standard case with an exponent equal to 1. However, for smaller $S$, the slope declines in absolute value, simply because the increase in variance produces mixing of smaller and larger firms.

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To summarize, network effects break Gibrat's law. Growth innovations are no longer iid and as a result the size distribution will not settle down. Our model produces aggregate dynamics as the size distribution $G(\cdot)$ changes over time, and the growth pdf changes along with it because the underlying network structure is changing. In the following sections we show that size distribution dynamics also feed back into time variation for the volatility distribution.

# H    Alternative Calibrations

To better understand the various parts of the model, we explore a number of alternative models to our benchmark model (M1). Table 11 lists the parameters. The results for size, volatility, and network moments are reported in Tables 12, 13, and 14, respectively.

Table 11: CALIBRATION

| Parameter | Description | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|---|
| 1. $N$ | # firms reported each period | 1000 | 1000 | 1000 | 1000 | 1000 |
| 2. $T$ | # years in simulation (after burn-in) | 1000 | 1000 | 1000 | 1000 | 1000 |
| 3. $\mu_g$ | Exogenous firm growth rate | 0 | 0 | 0 | 0 | 0 |
| 4. $\mu_{So}$ | Mean initial log size distribution | 10.20 | 10.20 | 10.20 | 10.20 | 10.20 |
| 5. $\sigma_{So}$ | Standard deviation initial log size distribution | 1.06 | 1.06 | 1.06 | 1.06 | 1.06 |
| 6. $\delta$ | Exogenous firm destruction rate | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 7. $\kappa$ | Governs the survival rate of connections | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| 8. $Z$ | Governs likelihood of new connections | 0.35 | 0.35 | 0.35 | 0.35 | 0.10 |
| 9. $\psi$ | Governs the importance of connections | 1 | 1 | 1 | 1 | 0.1 |
| 10. $\gamma$ | Importance of the network | 0.95 | 0.95 | 0.95 | 0.75 | 0.95 |
| 11. $\sigma$ | Fundamental shock volatility (i.i.d.) | 0.22 | 0.22 | 0.22 | 0.30 | 0.25 |
| 12. network | Full-degree or first-degree approximation | F | A | F | F | F |
| 13. $p(\cdot)$ | Governs probability of a connection | B | B | A | B | B |

*Notes:* In row 12, F indicates full-degree network and A indicates first degree-approximation. In row 13, B indicates benchmark and A indicates alternative assumption.

Model 2 (M2) is the first-degree approximation to the benchmark model, discussed in Section 3, with all parameters left unchanged. It produces dramatically different size and volatility results, underscoring the importance of network persistence. Out-degree Herfindahls in M2 are lower than in M1. Because of less concentrated networks, firms are better able to diversify the shocks that hit their customer network. Mean volatility (27%) and the dispersion in volatility across firms (26%) are substantially lower in the first-degree approximation as in the full-degree network. The median firm is 15% smaller while the largest firms are 34% smaller than in M1. In terms of macro moments, the size dispersion fluctuations almost completely disappear (4% time-series standard deviation versus 19%). The same is true for average firm volatility (3% time-series standard deviation versus 20%), aggregate growth volatility (4% versus 20%), and the volatility of the cross-sectional dispersion of volatility. The dramatic reduction in aggregate volatilities shows that higher-order connectivity amplifies shocks and is crucial in producing the patterns in aggregate moments we observe in the data.

Table 12: FIRM SIZE DISTRIBUTION

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) |
| | All | Top-33% | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|---|---|
| **Panel A: Cross-sectional Moments of Log Size** | | | | | | | |
| Avg | 9.61 | 11.63 | 11.63 | 11.47 | 11.66 | 11.78 | 11.57 |
| SD | 1.79 | 1.06 | 1.07 | 0.97 | 1.19 | 1.30 | 1.00 |
| 5% | 6.86 | 10.38 | 10.32 | 10.34 | 10.21 | 10.33 | 10.41 |
| 10% | 7.39 | 10.47 | 10.44 | 10.43 | 10.34 | 10.44 | 10.50 |
| 25% | 8.33 | 10.78 | 10.79 | 10.72 | 10.73 | 10.80 | 10.80 |
| Med | 9.48 | 11.39 | 11.42 | 11.26 | 11.43 | 11.47 | 11.34 |
| 75% | 10.77 | 12.25 | 12.25 | 11.99 | 12.35 | 12.43 | 12.08 |
| 90% | 12.05 | 13.10 | 13.10 | 12.78 | 13.30 | 13.54 | 12.91 |
| 95% | 12.76 | 13.64 | 13.68 | 13.34 | 13.93 | 14.32 | 13.51 |
| **Panel B: Time Series Properties of Size Distribution** | | | | | | | |
| SD of $\sigma_{S,t}$ | 0.24 | 0.14 | 0.19 | 0.04 | 0.43 | 0.11 | 0.15 |
| AR(1) $\sigma_{S,t}$ | 0.947 | 0.939 | 0.995 | 0.959 | 0.995 | 0.981 | 0.997 |
| SD of $\sigma_{g,t}$ | 0.15 | 0.11 | 0.05 | 0.01 | 0.01 | 0.02 | 0.03 |
| $H_t$ | 0.010 | 0.012 | 0.020 | 0.018 | 0.036 | 0.088 | 0.034 |

*Notes:* All reported moments in Panels A and B are time-series averages of the listed year-by-year cross sectional moments (cross section average, standard deviation, and percentiles) for the sample 1926–2010. The first column reports the full cross-section of firms . The second column reports results for the top-33% of firms in each year. Column 3 reports the corresponding moments for the benchmark model (M1). Column 4 shows the first-degree approximation of the benchmark model (M2). Columns 5-7 show three variations of the full-degree network model for different parameter values, described in the main text, and labeled M2-M5. Panel A reports moments of the log size distribution, where size is defined in the data as market equity. Panel B reports the time-series standard deviation and time-series persistence of size dispersion $\sigma_{S,t}$, defined as the cross-sectional standard deviation of log size, the time-series standard deviation of the cross section standard deviation of log growth rates $\sigma_{g,t}$ and the time-series average of the economy-wide Herfindahl index $H_t$.

Model 4 (M4) tells a similar story. Here, we revert to the full-degree network, but lower the persistence parameter of the network to $\gamma = 0.75$. Because lowering $\gamma$ lowers average volatility, we adjust the volatility of the shocks $\sigma$ upwards to 0.30. This model behaves similarly to M2 in that it has a more compact firm volatility distribution and much lower variability in size dispersion, mean variance, and variance dispersion than M1. Unlike M2, M4 has a much higher economy-wide Herfindahl index (0.088) indicating strong firm size concentration. This arises form the higher shock volatility which helps to produce large firms with many connections. The largest firms are too large compared to the data.

In Model 3 (M3), we modify the probability that a supplier is connected to a customer and make it less steep in size. Specifically, we change the probability of a connection from the benchmark equation (27) to the alternative

$$p_{i,j,t} = \frac{1}{Z + \log\left(\frac{\max_k\{S_{k,t}\}}{S_{i,t}}\right)}; \quad prob \ of \ conn = \max\{p_{i,j,t} + \kappa, 1\}$$

where we use the same scalar values for $Z$ and $\kappa$ than in the benchmark specification. This radically changes the average number of (untruncated) out-degrees from 10 in the benchmark to 441, while the *truncated* average out-degree actually falls. The median (truncated or untruncated) out-Herfindahl drops from 0.60 to 0.06, bringing it in line with the data. Because smaller firms now have bigger networks, they achieve better diversification. The dispersion of volatility is much lower (0.08 in M3 versus 0.45 in M1) and this dispersion fluctuates much less over time. The cross-sectional correlation between size and volatility becomes positive, but there is much less dispersion in volatility in the first place.

In Model 5 (M5) we consider a calibration that delivers a substantially lower out-degree Herfindahl than the benchmark model, bringing it closer to the data. We change the function $w_{i,j,t}$, which governs the importance of customer $j$ in supplier $i$'s network:

$$w_{i,j,t} = \frac{b_{i,j,t}S_{j,t}^{\psi}}{\sum_{k\neq i} b_{i,k,t}S_{k,t}^{\psi}} \quad \forall j \neq i$$

We make the importance of a given customer less steep in customer size by setting the curvature parameter $\psi = 0.1$. All other calibrations set $\psi = 1$. To further lower the out-degree Herfindahl, we lower $Z$ from its benchmark value of 0.35 to 0.10, making the probability of a connection steeper in supplier size. This generates more connections on average. Finally, we increase the volatility of the shocks to 25% so that the model generates the same mean variance as in the data. By giving large customers a smaller weight in the network, M5 reduces the median out-degree Herfindahl from 0.60 in M1 to 0.11; the truncated counterpart is 0.05 matching the data. The average number of (untruncated) connections is 32 and the median is 12. The larger number of connections helps to further lower the Herfindahls. M5 continues to generate substantial firm size dispersion and the right amount of average firm volatility, in part because of the higher shock volatility ($\sigma_\varepsilon = 0.25$ versus 0.22). However, the volatility dispersion (0.20) is severely curtailed, like in M3 (0.08), compared to the 0.45 in M1 and the even higher numbers in the data. M5 cannot generate enough highly volatile firms, compared to the data. Aggregate moments such as size dispersion, mean variance, variance dispersion, and aggregate growth display meaningful time variation, but less than in the benchmark model. Despite this drawback, this model has arguably a more realistic network structure, and continues to generate interesting variation in the key moments of interest.
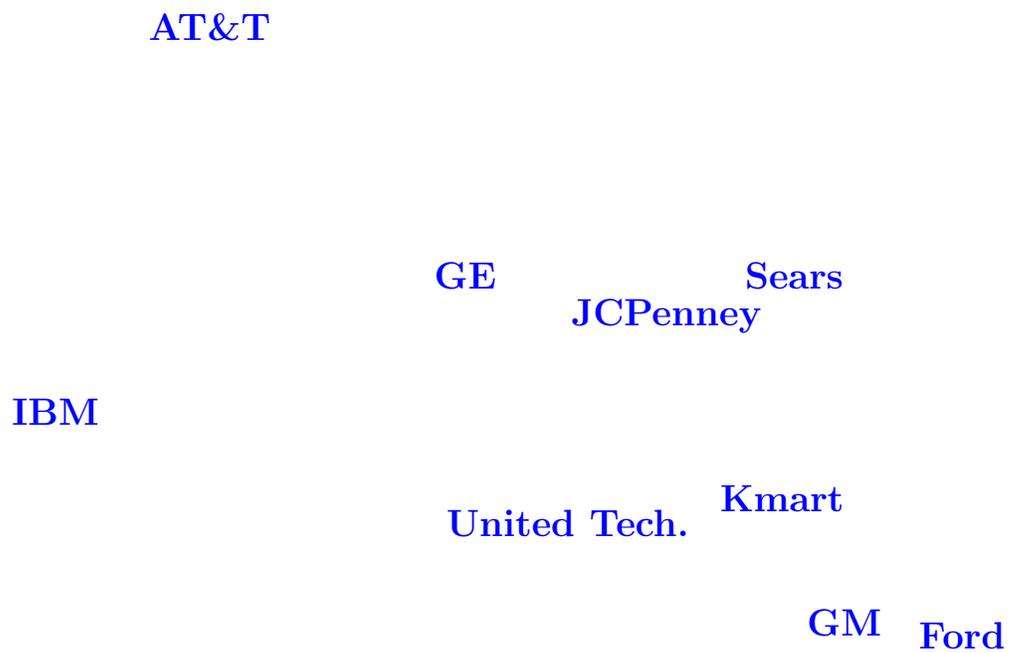
Table 13: Firm Volatility Distribution

| | (1)<br>All<br>Returns | (2)<br>Top-33%<br>Returns | (3)<br>All<br>Sales | (4)<br>Top-33%<br>Sales | (5)<br>M1 | (6)<br>M2 | (7)<br>M3 | (8)<br>M4 | (9)<br>M5 |
|---|---|---|---|---|---|---|---|---|---|
| **Panel A: Cross-Sectional Moments of Firm Volatility** | | | | | | | | | |
| Avg | 0.40 | 0.30 | 0.29 | 0.24 | 0.37 | 0.27 | 0.51 | 0.40 | 0.31 |
| SD | 0.96 | 0.73 | 1.51 | 1.42 | 0.45 | 0.26 | 0.08 | 0.29 | 0.20 |
| 5% | 0.19 | 0.17 | 0.09 | 0.08 | 0.26 | 0.22 | 0.48 | 0.31 | 0.27 |
| 10% | 0.22 | 0.19 | 0.11 | 0.10 | 0.28 | 0.23 | 0.48 | 0.33 | 0.27 |
| 25% | 0.29 | 0.24 | 0.17 | 0.14 | 0.31 | 0.24 | 0.49 | 0.36 | 0.29 |
| Med | 0.40 | 0.30 | 0.29 | 0.23 | 0.36 | 0.27 | 0.50 | 0.40 | 0.30 |
| 75% | 0.56 | 0.38 | 0.48 | 0.38 | 0.42 | 0.29 | 0.52 | 0.44 | 0.32 |
| 90% | 0.75 | 0.48 | 0.77 | 0.60 | 0.49 | 0.32 | 0.53 | 0.48 | 0.35 |
| 95% | 0.90 | 0.54 | 1.05 | 0.77 | 0.54 | 0.33 | 0.54 | 0.50 | 0.36 |
| **Panel B: Cross-Sectional Moments of Size-Volatility Distribution** | | | | | | | | | |
| $Corr(S_t, V_{t+1})$ | $-0.57$ | $-0.33$ | $-0.33$ | $-0.18$ | $-0.42$ | $-0.49$ | 0.20 | $-0.50$ | $-0.37$ |
| $\beta(S_t, V_{t+1})$ | $-0.32$ | $-0.23$ | $-0.27$ | $-0.22$ | $-0.18$ | $-0.13$ | 0.01 | $-0.11$ | $-0.07$ |
| **Panel C: Time Series Properties of Volatility Distribution** | | | | | | | | | |
| SD of $\mu_{\sigma^2,t}$ | 0.69 | 0.64 | 0.46 | 0.42 | 0.20 | 0.03 | 0.82 | 0.09 | 0.15 |
| SD of $\sigma_{\sigma^2,t}$ | 0.18 | 0.12 | 0.13 | 0.13 | 0.05 | 0.01 | 0.02 | 0.01 | 0.07 |
| $Corr(\sigma_{S,t}, \mu_{\sigma^2,t})$ | 0.71 | 0.55 | 0.51 | 0.51 | 0.95 | 0.86 | 0.76 | 0.90 | 0.93 |
| $Corr(\sigma_{S,t}, \sigma_{\sigma^2,t})$ | 0.76 | 0.76 | 0.57 | 0.38 | 0.77 | 0.09 | 0.34 | $-0.54$ | 0.94 |
| SD of $g_{agg,t}$ | 0.21 | 0.20 | 0.21 | 0.20 | 0.20 | 0.04 | 0.62 | 0.15 | 0.15 |
| $Corr(g_{agg,t}, \sigma_{S,t})$ | $-0.06$ | $-0.08$ | $-0.06$ | $-0.08$ | $-0.02$ | 0.02 | $-0.13$ | 0.02 | $-0.01$ |
| $Corr(g_{agg,t}, \mu_{\sigma^2,t})$ | $-0.32$ | $-0.43$ | 0.01 | $-0.00$ | $-0.01$ | $-0.01$ | $-0.11$ | 0.01 | $-0.01$ |
| $Corr(g_{agg,t}, \sigma_{\sigma^2,t})$ | 0.04 | $-0.19$ | 0.18 | 0.14 | 0.02 | 0.04 | $-0.03$ | 0.01 | $-0.00$ |

*Notes:* All reported moments in Panels A and B are time-series averages of the listed year-by-year cross sectional moments (cross section average, standard deviation, and percentiles) for the sample 1926–2010. The first and third columns report return volatility and sales growth volatility, respectively, for the full cross-section of firms. The second and fourth columns report results for the largest 33% of firms by market capitalization in each year. Annual variances in Columns 1 and 2 are calculated from daily stock return data, while variances in columns 3 and 4 are based on 20 quarters of sales growth rates (in the current and the next four years). Moments are computed based on the log variance distribution, but are expressed as volatilities in levels for exposition. That is, we exponentiate each moment of the log distribution and take the square root. Column 5 reports the corresponding moments for the benchmark model (M1). Column 6 shows the first-degree approximation of the benchmark model (M2). Columns 7-9 show three variations of the full-degree network model for different parameter values, described in the main text. Log variances in the model are constructed with added estimation noise to make them comparable to the moments in the data.

Table 14: NETWORK MOMENTS

| | (1) All Returns | (2) Top-33% Returns | (3) All Sales | (4) Top-33% Sales | (5) M1 | (6) M2 | (7) M3 | (8) M4 | (9) M5 |
|---|---|---|---|---|---|---|---|---|---|
| **Panel A: Out-degree Moments** | | | | | | | | | |
| median $K^{out}$ | 1.00 | 1.00 | – | – | 1.71 | 1.99 | 0.75 | 1.12 | 1.03 |
| $99^{th}$ % $K^{out}$ | 3.38 | 3.32 | – | – | 4.66 | 4.94 | 2.24 | 4.05 | 7.93 |
| median $H^{out}$ | 0.05 | 0.04 | – | – | 0.60 | 0.51 | 0.04 | 0.88 | 0.05 |
| $99^{th}$ % $H^{out}$ | 0.95 | 0.54 | – | – | 1.00 | 1.00 | 0.23 | 1.00 | 0.72 |
| $Corr(K_t^{out}, S_t)$ | 0.01 | −0.07 | – | – | 0.34 | 0.25 | −0.08 | 0.45 | −0.55 |
| $Corr(H_t^{out}, S_t)$ | −0.30 | −0.09 | – | – | −0.71 | −0.67 | −0.12 | −0.72 | −0.53 |
| $Corr(H_t^{out}, V_{t+1})$ | 0.17 | 0.11 | 0.25 | 0.09 | 0.54 | 0.70 | 0.45 | 0.63 | 0.58 |
| **Panel B: In-degree Moments** | | | | | | | | | |
| median $K^{in}$ | 1.00 | 1.12 | – | – | 1.90 | 2.00 | 0.00 | 1.35 | 2.06 |
| $99^{th}$ % $K^{in}$ | 31.61 | 16.80 | – | – | 7.39 | 7.42 | 45.41 | 5.90 | 6.93 |
| median $H^{in}$ | 0.00 | 0.00 | – | – | 0.11 | 0.11 | 0.00 | 0.07 | 0.01 |
| $99^{th}$ % $H^{in}$ | 0.28 | 0.24 | – | – | 0.85 | 0.83 | 0.00 | 0.76 | 0.05 |
| $Corr(K_t^{in}, S_t)$ | 0.37 | 0.49 | – | – | 0.49 | 0.46 | 0.31 | 0.38 | 0.16 |
| $Corr(H_t^{in}, S_t)$ | −0.44 | −0.28 | – | – | 0.03 | 0.03 | 0.21 | −0.07 | 0.01 |
| $Corr(H_t^{in}, V_{t+1})$ | 0.25 | 0.19 | 0.09 | 0.12 | 0.01 | −0.00 | 0.17 | 0.05 | 0.03 |

*Notes:* All reported moments are time-series averages unless explicitly mentioned otherwise. The first and second columns report data for the cross-section of firms for which we have customer information from Compustat. The sample is 1980-2009. The variance in Column 1 is calculated based on daily stock return data, while the variance in Column 2 is based on 20 quarters worth of annual sales growth (in the current and the next four years). Column 3 reports the corresponding moments for the benchmark model (M1). Column 4 shows the first-degree approximation of the benchmark model (M2). Columns 5-7 show three variations of the full-degree network model for different parameter values, described in the main text. Log variances in the model are constructed with added estimation noise to make them comparable to the moments in the data.

**AT&T**

**GE** **Sears**
**JCPenney**

**IBM**

**United Tech.** **Kmart**

**GM** **Ford**

*Notes:* The figure shows one region of the Compustat customer-supplier network for 1980. Size of nodes represent log sales of a firm, and arrows represent directed links from supplier to customer (with arrow thickness corresponding to the weight of the link). We highlight certain central nodes including AT&T, IBM, United Technologies, General Electric, Sears, JCPenney, General Motors and Ford.

Figure 12: Variance of Log Employment: Compustat vs. Census Data
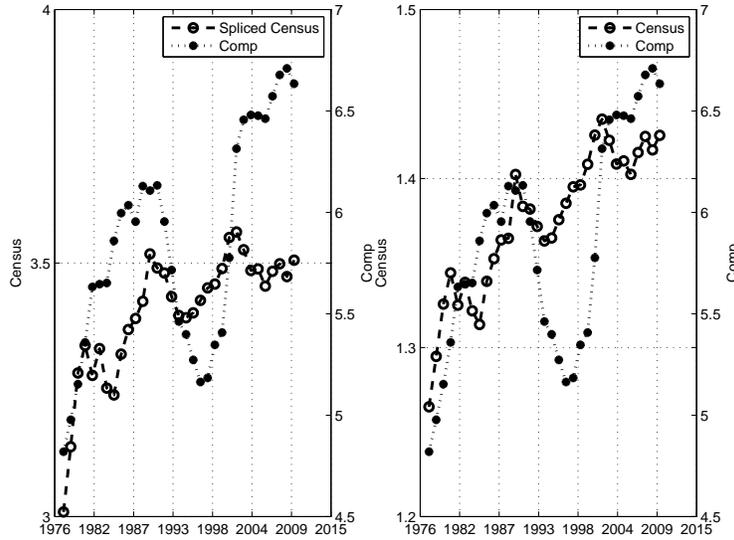


*Notes:* The figure plots the cross-sectional variance of log employment in the Compustat and Census data (left-hand-side axis). The data is annual and the sample covers 1977-2009. Source: U.S. Small Business Administration, Office of Advocacy, from data provided by the U.S. Census Bureau, Business Dynamics Statistics.

Figure 13: Firm Dynamics
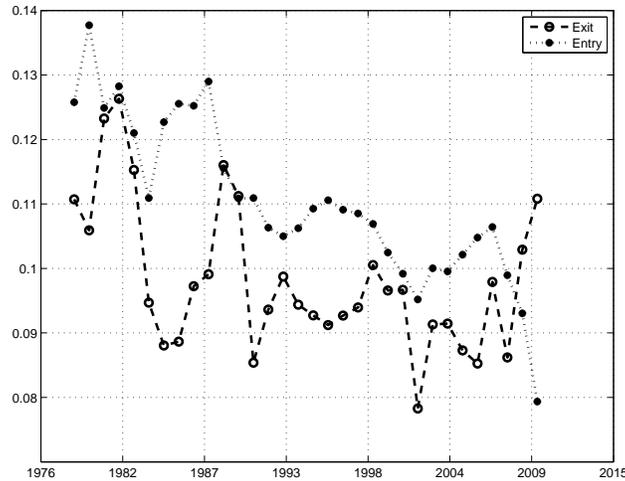


*Notes:* The figure plots the entry and exit rates for all U.S. firms based on Census data. The data is annual and the sample covers 1977-2009. Source: U.S. Small Business Administration, Office of Advocacy, from data provided by the U.S. Census Bureau, Business Dynamics Statistics.

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