Using Money Signals to Improve Taylor Rule Performance in the New Keynesian Model

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I. Introduction:

A social welfare function from Clarida, Gali and Gertler (1999)

\[ W = -\frac{1}{2} \sum_{i=1}^{\infty} \beta_i E_t [\pi_{t+i}^2 + \Gamma x_{t+i}^2] \]

was introduced for an approximation of monetary policy objective function to the utility of representative households.\(^1\) This function can be interpreted as a quadratic loss function

\[ \mathcal{L} = \frac{1}{2} \sum_{i=1}^{\infty} \beta_i E_t [\pi_{t+i}^2 + \Gamma x_{t+i}^2] \]

under “the assumption that the Fed’s objective is to minimize a social welfare loss function, defined in terms of output gap and inflation (Walsh 2003, p.523).” Assume also that the Fed sets policy in each period before observing the current shocks to the output gap and inflation, and “assume that information on money, but not output and inflation, is immediately available (Walsh 2003, p.432).” The loss function \( \mathcal{L} \) can be treated as a criterion for evaluating the policy rules in each period and be simplified as

\[ (1) \quad L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2]. \]

I then use equation (1) to evaluate the performance of a Taylor rule in the baseline new Keynesian model which the rule and the model are expressed in three equations, an

\(^1\) For the derivation of this social welfare function, see Michael Woodford (2001a;2003), and Rotemberg and Woodford (1997). For a welfare-base evaluation of policy rules, see Rotemberg and Woodford (1999).
interest rate equation

\[ i_t = \varphi_x x_t + \varphi_\pi \pi_t , \]

a new Keynesian IS equation

\[ x_t = E_t[x_{t+1}] - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}]) + g_t , \]

and a new Keynesian Philips curve

\[ \pi_t = k x_t + \beta E_t[\pi_{t+1}] + u_t . \]

Since the information content of output and inflation do not include current shocks, it is equivalent to say that the Fed adopt an expected standard Taylor rule as

\[ \hat{i}_t = \varphi_x E_{t-1}[x_t] + \varphi_\pi E_{t-1}[\pi_t] \]

which implies that the policy rule cannot directly response to current shocks, in this environment, “it will be impossible to determine the source of shocks that have caused interest rate to move (Walsh 2003, p.432).” This causes higher period social welfare losses, described by equation (1), than its minimum value when the Fed has a full-information Taylor rule. In section IV, I will provide an illustration for the performance of these two policy rules by comparing their social welfare losses associated with the model economy for output gap and inflation described by the baseline new Keynesian model in the chapter 1.

I then propose a combination policy rule in the spirit of Poole (1970) which the Fed should use money signals to determine the source of shocks that have also caused money to fluctuate. Poole (1970) in his classic analysis for showing “how the stochastic structure of the economy—the nature and relative importance of different types of
disturbances—would determine the optimal instrument (Walsh 2003, p.431)” shed light
on using equation (1) for assessing the proposed policy rule. If the social welfare loss is
improved by the proposed policy rule, the signals from money will be proved to be useful
for the Fed’s monetary policy same reason as Poole (1970) determine the optimal
instrument between an interest rate or a monetary aggregate.

The proposed policy rule combines an expected standard Taylor rule for an interest
rate and the money as signal for unobserved shocks. I will show later in section V that
this combination policy rule will improve the period social welfare losses.

I start from studying the relationship between the social welfare losses and the
standard Taylor rule in the section II. The standard Taylor rule assumes that the
information on output and inflation is immediately available, but not directly been set in
response to current output shocks. Although the standard Taylor rule has the coefficients,
determining the response of the Fed to fluctuations in output and inflation, same as a
Taylor rule with direct response to current output shocks, the standard Taylor rule only
has a limit for the optimal solution of social welfare losses.

In section III, I will show the relationship between an interest rate and the
movement of money. This relationship provides the reasoning for constructing the
proposed combination policy rule. In section VI, I will show how the Fed infer
unobserved shocks from the money as signal as well as its relationship with the
disturbance of money demand. Section VII provides an illustration that how the
disturbance of money demand affects the social welfare losses of the proposed policy rule.
II. The Period Social Welfare Losses by Having The Standard Taylor Rule:

Let the standard Taylor rule be expressed as

\[ i_t = \varphi_x x_t + \varphi_{\pi} \pi_t. \]

Clarida, Gali and Gertler (1999) derived the optimality condition: \( x_t = -\frac{k}{\Gamma} \pi_t \) and showed that a corresponding interest rate can be obtained by using this condition and a new Keynesian IS curve with the assumption of rationally expected output gap and inflation. Their corresponding interest rate, which has a direct response to IS-shocks \( g_t \) in it, is not necessary an optimal Taylor rule because they did not have a Taylor-type rule in their social welfare function.

Woodford (2001b) showed an empirical result that it is practical to improve welfare by having a Taylor rule directly\(^2\) in response to the IS-shock \( g_t \). His proposed Taylor rule is optimal given the satisfaction of the determinacy condition of new Keynesian model for the analysis of monetary policy.

\(^2\) Woodford (2001b, p.235): “...the rule must include a time-varying intercept \( \sigma (g_t + \mathbb{E}_t x_{t+1}) \) for consistency with a stable inflation rate and output gap.”
Thurston (2010/2012) then provided analytical derivation, in general, that an optimal Taylor rule should include the direct response to the IS-shocks $g_t$ when $g_t$ exists in the NKМ. Thurston’s optimal Taylor rule also is guaranteed to meet the determinacy condition of the NKМ. How much the Fed should adjust interest rate in response to $g_t$ depends on the coefficient $\varphi_g$, whose optimal value is equal to $\sigma$. He further argued that since $g_t$ is already in the NKМ, we need a $\varphi_g$ in the Taylor rule unless there some reason we cannot. If we cannot, we have a constrained optimum problem when $g_t$ exist.

We can view the standard Taylor rule as having $\varphi_g = 0$ when a Taylor rule was written as

$$i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t$$

in Thurston (2012). The standard Taylor rule implies that the interest rate does not directly response to $g$ at period $t$. This is a special case where $g$ exists but $\varphi_g = 0$. I will show later in this section that the constraint on the standard Taylor rule’s coefficients $\varphi_x$ and $\varphi_\pi$ is the same as the constraint on the optimal Taylor rule’s coefficients $\varphi_x$ and $\varphi_\pi$ with $\varphi_g$ in Thurston (2012). However, this same constraint of $\varphi_x$ and $\varphi_\pi$ on the standard Taylor rule cannot guarantee for having the same optimal paths of $x_t$ and $\pi_t$ of the Taylor rule with $\varphi_g$. Furthermore, the minimum social welfare loss (denoted as $L_1$) using the standard Taylor rule may be bigger and never be smaller than the minimum social welfare loss (denoted as $L_0$) by using the optimal Taylor rule with $\varphi_g$. The only possibility for $L_1 = L_0$ is when the values of $\varphi$’s approach to infinity given the constraint on them held.
Using the general solutions of $x_t$ and $\pi_t$ inspired by and attributed to Thurston (2010/2012)$^3$, which is the first paper about the general solutions of the new Keynesian model, and the period losses $L$ from equation (1) as

$$L = \frac{1}{2}E(\pi_t^2 + \Gamma x_t^2),$$

I obtain the first order conditions for the minimum value of $L$ by differentiating $L$ with respect to $\phi_x$ and $\phi_\pi$ for the reason that the general solutions of $x_t$ and $\pi_t$ are the functions of $\phi_x$ and $\phi_\pi$. These first order conditions are the constraint on the standard Taylor rule’s coefficients $\phi_x$ and $\phi_\pi$ which is expressed as

$$\phi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)}\phi_x + \frac{k\sigma(\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho.$$

Substituting the constraint into $L$, I obtain the minimum social welfare loss $L_1$,

$$L_1 = \frac{1}{2}\left(\Gamma \frac{\Gamma}{(\beta \rho - 1)^2 + k^2}\right)VAR_u + \frac{1}{2}\left(\frac{\Gamma(\beta \lambda - 1)^2 + k^2}{D^2}\right)VAR_g,$$

where $D = k\left(-\frac{k}{\Gamma(\beta \rho - 1)}\phi_x + \frac{k\sigma(\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho\right) + (1 - \beta \lambda)\phi_x + (\lambda - 1)(\beta \lambda - 1)\sigma - k\rho$.

$VAR_u$ and $VAR_g$ are the variance of inflation and the variance of output gap. Comparing $L_1$ with $L_0$, which the expression is as

$$L_0 = \frac{1}{2}\left(\frac{\Gamma}{(\beta \rho - 1)^2 + k^2}\right)VAR_u,$$

it indicates that the $VAR_u$ terms will be minimized for all combinations of $\phi_x$ and $\phi_\pi$ that

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$^3$ This general solutions of $x_t$ and $\pi_t$ are derived by having the standard Taylor rule in the associated NKM. Appendix A shows the details of the derivations. Since the standard Taylor rule does not have $\phi_g$, this general solutions of $x_t$ and $\pi_t$ are different from Thurston’s general solutions. These differences do not change the conclusion later presented in this paper.
satisfy the constraint $\varphi_{\pi} = -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k\sigma(\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho$. When $\varphi_g = 0$, there will be a $VARg$ term in $L$ as shown in $L1$. When $\varphi_g = \sigma$, the $VARg$ term is eliminated.

If we differentiate $D$ with respect to $\varphi_x$ and $\varphi_\pi$, we can observe that $D$ is positively related to $\varphi_x$ and $\varphi_\pi$, i.e., $\frac{d}{dD} \varphi_x > 0$ and $\frac{d}{dD} \varphi_\pi > 0$. This implies that the $VARg$ term declines in both $\varphi_x$ and $\varphi_\pi$. When both $\varphi_x$ and $\varphi_\pi$ approach to infinity, the $VARg$ term approaches to zero. Jointly, when the constraint on $\varphi's$ is met and $\varphi's$ are approaching to infinity, $L1$ is approaching $L0$. Figure 1 illustrates this result with $k = 0.3, \beta = 0.99, \Gamma = 2, \sigma = 1, \lambda = 0.5, \rho = 0.5, VARg = 1$ and $VARu = 1$. 

![Figure 1: L1 Approaches L0](image)
There is none, only a limit for the optimum solution of social welfare when a Taylor rule does not have $\varphi_g$. Although an unconstrained solution does not exist, there may be a constrained solution which is not necessarily defined in the model. Appendix B shows this constrained solution.

The standard Taylor rule case discussed above is very important when the white noises $\varepsilon_t$ in $g_t$ and $\eta_t$ in $u_t$ are unobserved. The standard Taylor rule case implies that, although the Fed does not directly response to unobserved white noises in the new Keynesian model, this rule will have the same constraint on $\varphi_x$ and $\varphi_{\pi}$ for the optimum social welfare loss. In other words, $\varphi_g = 0$ implies that the rule cannot directly make adjustment to interest rates in response to $g_t$. This is consistent with the assumption that information on output and inflation are not immediately available for the Fed to make adjustment in interest rates from its policy rule.

On the other hand, information on money is immediately available for the Fed. It is straightforward to use the movement of money for determining the sources of shocks. In next section, I will show the reasoning of applying money in an interest rate policy rule by deriving the relationship between an interest rate and the movement of money.

**III. The Interest Rates and The Movement of Money:**

The relationship between interest rates and the demand for money helps us to understand how to use the money as signal (hereafter, the MAS) for unobserved $\varepsilon_t$ and $\eta_t$ when all of the expected values are locked in with t-1 information. Using the Euler
condition of marginal rate of substitution between holding money and consumption:

\[
\frac{\left(\frac{M_t}{P_t}\right)^{-b}}{e^{-\sigma}} = \frac{i_t}{1+i_t},
\]
we can derive an equation for money demand:

\[
m_t - p_t = \frac{\sigma}{b} y_t - \frac{1}{b} i_t,
\]

where \(b\) is the interest rate elasticity of money demand and the value of \(b\) must be greater than zero. I then obtain the expression of interest rate by rearranging this money demand as

\[
i_t = \sigma y_t + bp_t - bm_t.
\]

Suppose the money demand is also an AR(1) process which

\[
m_t = \zeta m_{t-1} + v_t
\]

is a white noise. This \(v\) is therefore assumed to contain the current information of unobserved \(\varepsilon_t\) and \(\eta_t\). I will construct the specification of \(v\) in the new Keynesian model economy later in section V. And if \(y_t\) and \(p_t\) are at their steady state denoted as \(\bar{y}_t^4\) and \(\bar{p}_t\), the above expression of interest rate can be rewritten as

\[
i_t = \sigma \bar{y}_t + b \bar{p}_t - bm_t^*.
\]

Now let the expected interest rate is denoted as

\[
\hat{i}_t = E_{t-1}[i_t]
\]

, which implies that

\[
\hat{i}_t = \sigma \bar{y}_t + b \bar{p}_t - bE_{t-1}[m_t^*].
\]

Then the value difference between actual interest rate and expected interest rate is negatively related to the value difference between actual money demand and expected

\[\text{\textsuperscript{4} In this chapter, I assume } \bar{y}_t \text{ is the output at full-employment level.}\]
money demand:

\[ i_t - \hat{i}_t = -b(m_t^* - E_{t-1}[m_t^*]), \]

which implies that the value difference between actual interest rate and expected interest rate is the white noise of the money process, \( v_t \). This equality of interest rate and money demand shows the relation between interest rate and money movement. The changes in interest rate are negatively proportional to the changes in money demand. This equality can be rearranged by moving the expected interest rate to the right-hand side of the equation as:

\[ (5) \quad i_t = \hat{i}_t - b(m_t^* - E_{t-1}[m_t^*]). \]

I then obtain the relationship where the actual interest rate is expressed in terms of the expected interest rate and the changes in money demand which is the MAS. Later in section V, I will show how to use this relationship for constructing a combination policy rule for improving monetary policy from the \( m_t^* \) terms.

Before I start applying the MAS for the Fed’s policy rule, it is equally important to know how much the difference of social welfare losses between the standard Taylor rule and the expected Taylor rule. In the next section, I will compare this losses difference analytically.

When the Fed sets interest rate before it observes current changes in output gap \(x_t\) and inflation \(\pi_t\), this implies that the Fed has an expected standard Taylor rule expressed as

\[
\hat{\tau}_t = \hat{\phi}_x E_{t-1}[x_t] + \hat{\phi}_\pi E_{t-1}[\pi_t].
\]

This rule uses expected values of output gap and inflation instead of using the current values of them; therefore, the new information \(\varepsilon_t\) and \(\eta_t\) will not be observed by the Fed. Since all of the expected values are locked in with \(t-1\) information, the only new shocks are unobserved \(\varepsilon_t\) and \(\eta_t\). The Fed will deliberately but temporarily violate the standard Taylor rule because of this new information in current period. In other words, even if the constraint on \(\hat{\phi}'s\) for the expected standard Taylor rule is the same as the constraint on \(\phi's\) for the standard Taylor rule, the social welfare loss of using the expected standard Taylor rule will be bigger than using the standard Taylor rule.

Let \(\hat{x}_t\) and \(\hat{\pi}_t\) denote the output gap and inflation with the expected standard Taylor rule. Then the difference between output gap and inflation using the standard Taylor rule and the expected standard Taylor rule are denoted as

\[
\begin{align*}
\Delta x_t &= x_t - \hat{x}_t = \varepsilon_t \\
\Delta \pi_t &= \pi_t - \hat{\pi}_t = k\varepsilon_t + \eta_t.
\end{align*}
\]

This difference increases the social welfare losses by the amounts of

\[
\Delta L_t = \frac{1}{2}(k^2 + \Gamma)VARg + \frac{1}{2}VARu. \quad (7)
\]

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5 The shocks in output gap and inflation are assumed to be AR(1) shocks so \(g_t = \lambda g_{t-1} + \varepsilon_t\) and \(u_t = \rho u_{t-1} + \eta_t\).

6 \(\Delta L_t = \frac{1}{2}E[(\Delta \pi_t)^2 + \Gamma(\Delta x_t)^2] = \frac{1}{2}[E(k\varepsilon_t + \eta_t)^2 + \Gamma E(\varepsilon_t)^2] = \frac{1}{2}(k^2 + \Gamma)VARg + \frac{1}{2}VARu.\)
Figure 2 shows this difference as $\phi$'s are approaching to infinity by using the same parameter values of the figure 1. The social welfare loss with the expected standard Taylor rule ($L_2$) is always bigger than the social welfare loss with the standard Taylor rule ($L_1$) and this difference cannot be eliminated as $\phi$'s are approaching to infinity.

It is straightforward to reduce the difference of social welfare losses $\Delta L_t$, described in equation (7), by combining the MAS and the expected standard Taylor rule. In the next section, I will show how to use the MAS derived in the previous section to reduce $\Delta L_t$, which is the improvement of monetary policy.

V. The Expected Standard Taylor Rule and The Money as Signal
It is equivalent to say that the improvement of the Fed’s policy rule is to reduce $\Delta L_t$ by using a combination policy rule—combining an expected standard Taylor rule and the MAS—when $\epsilon_t$ and $\eta_t$ are unobserved by the Fed. From previous two sections we learn that (1) the standard Taylor rule equals the expected standard Taylor rule and the money as signal (MAS) in the section III, and (2) the social welfare loss with the expected standard Taylor rule is bigger than the social welfare loss with the standard Taylor rule in the section IV.

This difference of social welfare losses cannot be reduced by having $\phi'$s approach to infinity. Thus using the MAS is the simple and direct way to reduce the difference of social welfare losses, i.e., $\Delta W$, when new information $\epsilon_t$ and $\eta_t$ are unobserved. The MAS is actually a linear relation of $\epsilon_t$ and $\eta_t$. To see this relation, first by letting $x_t = y_t - y_t'$ and $\pi_t = p_t - p_{t-1}$, I can rewrite the money demand condition $m_t - p_t = \frac{\sigma}{b} y_t - \frac{1}{b} i_t + \omega_t$ as

$$m_t = \pi_t + p_{t-1} + \frac{\sigma}{b} (x_t + y_t') - \frac{1}{b} i_t + \omega_t$$

$$= \hat{\pi}_t + \Delta \pi_t + p_{t-1} + \frac{\sigma}{b} (\hat{x}_t + \Delta x_t + y_t') - \frac{1}{b} i_t + \omega_t$$

, where $y_t'$ is the full-employment level of output and $\omega_t$ is the money demand disturbance.\(^7\) Then the value difference between actual money and expected money, which is the MAS, is

\[^7\text{Note that } \pi_t = \hat{\pi}_t + \Delta \pi_t \text{ and } x_t = \hat{x}_t + \Delta x_t \text{ from section IV.}\]
\[
\begin{align*}
m_t &- E_{t-1}[m_t] = \Delta \pi_t + \frac{\sigma}{b}(\Delta x_t) + \omega_t \\
&= k\varepsilon_t + \eta_t + \frac{\sigma}{b}\varepsilon_t + \omega_t \\
&= \left(k + \frac{\sigma}{b}\right)\varepsilon_t + \eta_t + \omega_t.
\end{align*}
\]

Then from the section III, the actual interest rate rule is the expected interest rate and the MAS. I can rewrite the actual Taylor rule by adding the right hand side of the equation of the MAS into the expected standard Taylor rule, which can be shown as

\[(8) \quad i_t = \phi_x E_{t-1}[x_t] + \phi_{\pi} E_{t-1}[\pi_t] + \phi_m \left(\left(k + \frac{\sigma}{b}\right)\varepsilon_t + \eta_t + \omega_t\right).\]

Replacing \(i_t\) for the standard Taylor rule in the baseline NKM and based on what we have learned in the section III, the constraint on \(\phi_x\) and \(\phi_{\pi}\) for the social welfare loss\(^8\) is the same. Next, I obtain the analytical value of \(\phi_m\) by using the same steps of obtaining the constraint of \(\phi_x\) and \(\phi_{\pi}\) for optimal welfare loss using the first order condition with respect to \(\phi_m\), which is expressed as:

\[(9) \quad \phi_m = \frac{\sigma b (bk + \sigma) VARg}{(bk + \sigma)^2 VARg + b^2 VARu + b^2 VARm},\]

where \(VARg\) is the variance of output gap shocks \(\varepsilon_t\), \(VARu\) is the variance of inflation shocks \(\eta_t\) and \(VARm\) is the variance of money demand shocks \(\omega_t\).

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\(^8\) This social welfare loss \(L3\) with \(i_t\) is \(L2 + \frac{1}{2} \left(1 + \frac{(k^2 + \Gamma)^2}{\sigma^2} \phi_m^2\right) VARu + \frac{1}{2} \left(\frac{(k^2 + \Gamma)(bk + \sigma)\phi_m - \sigma\phi_m^2}{\sigma^2 b^2}\right) VARg + \frac{1}{2} \left(\frac{(k^2 + \Gamma)^2}{\sigma^2} \phi_m^2\right) VARm.\)
Equation (9) inherits the spirit of Poole (1970). When the value of $VARm$ is very large so the value of $\phi_m$ is very close to zero, in this situation, the MAS cannot be an effective tool for determining the sources of shocks. The validity of equation (9) can be examined by letting $VARu = 0$ and $VARm = 0$, and $\Delta L_t$ from the equation (7) will be zero when I have the equation (9) in the equation (8). This implies that the MAS is able to eliminate the difference in the social welfare losses which caused by unobserved $\varepsilon_t$. A similar testing result also applies to the case when only unobserved $\eta_t$ happens in the model economy.

Figure 3 shows the improvement of social welfare loss from the MAS. $L_3$ is the social welfare loss with the MAS when the noise of money demand exists. $L_4$ is the social welfare loss with the MAS when the noise in money demand does not exist. Both $L_3$ and $L_4$ are smaller than $L_2$ which is the social welfare loss with the expected standard Taylor rule described in section III.

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9 According to Poole (1970), the Fed should use an interest rate as instrument when the social welfare losses caused by the variances of shocks under the interest rate operating procedure are smaller than under the money-supply operating procedure.

10 $b$ is assumed with value equal to 0.5. All the other parameter values are the same as used in section III.

11 The variance of money demand $\sigma_m^2$ equals to one.

12 The variance of money demand $\sigma_m^2$ equals to zero.
It is clear from Figure 3 that the social welfare loss cannot be improved by letting $\varphi's$ approach to infinity. The only way to improve the social welfare loss using the expected standard Taylor rule is applying the MAS for reducing the impacts of unobserved $\epsilon_t$ and $\eta_t$.

For simplicity, a special case for the $\varphi_m$ can be also shown as

$$\varphi_m = \frac{\sigma b(bk + \sigma)}{(bk + \sigma)^2 + 2b^2}$$

when the disturbances of output gap, inflation and money are i.i.d standard normal.
distributions, so \( VARG = VARu = VARm = 1 \). Equation (10) is a constant for optimal social welfare losses. However, Figure C in the Appendix C shows that \( \phi_m \) is always greater than zero and has a limit when \( b \) approaches infinity by using equation (10). The long time debate on the stability of money demand through the different estimated values of interest rate elasticity of money demand \( b \) should not affect the effectiveness of \( \phi_m \).

### VI. The Poole Approach and the Money as Signal

How does the Fed infer unobserved \( \epsilon_t \) and \( \eta_t \) from the MAS? The Fed controls the money supply so it is capable of calculating the MAS. Recalled that

\[
\begin{align*}
\left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t 
\end{align*}
\]

The information content of the MAS is then used to indicate \( \epsilon_t \) and \( \eta_t \) as followed. The estimated coefficient of \( \hat{\epsilon}_t \) conditional on

\[
\left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t
\]

is that

\[\phi_m\]

The value of \( \phi_m \) is determined by the inverse of the interest rate elasticity \( b \), which is the factor for the stability of money demand. \( \phi_m \) will equal to zero only when \( b = 0 \) or \( \frac{\sigma}{k} = 0 \). Since \( b > 0 \) from our derivation of money demand in section III, \( \phi_m > 0 \). Teles and Zhou (2005, p.52), “Ball (2001) argues that the data after 1987 represent evidence against a stable money demand. He estimates a linear relationship between logarithm of real money, the logarithm of output, and a nominal interest rate for subperiods of 1903-94. For the period 1903-87 the evidence is consistent with a stable relationship with a unitary income elasticity and a relatively high interest elasticity, as shown by Lucas (1988) and Stock and Watson (1993). However, the need to account for the low reaction of M1 to lower interest rates and higher output after 1980 lowers both the estimated interest elasticity and income elasticity. The relatively low income and interest elasticity in the postwar period (1974-94) are significantly different from the unitary income elasticity and relatively high interest elasticity in the prewar period (1903-45), leading Ball to argue against a stable long run money demand.”

Walsh (2010, pp. 48-52)
\[ \hat{\epsilon}_t = \hat{\alpha}_1 \epsilon_t, \]
\[ \hat{\alpha}_1 = \text{Cov} \left[ \epsilon_t, \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \right] / \text{Var} \left[ \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \right], \]
\[ \hat{\alpha}_1 = \left( k + \frac{\sigma}{b} \right) \sigma^2_g / \left[ \left( k + \frac{\sigma}{b} \right)^2 \sigma^2_g + \sigma^2_u + \sigma^2_m \right], \]

and the estimated coefficient of \( \hat{\eta}_t \) conditional on \( \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \) is that
\[ \hat{\eta}_t = \hat{\alpha}_2 \eta_t, \]
\[ \hat{\alpha}_2 = \text{Cov} \left[ \eta_t, \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \right] / \text{Var} \left[ \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \right], \]
\[ \hat{\alpha}_2 = \sigma^2_u / \left[ \left( k + \frac{\sigma}{b} \right)^2 \sigma^2_g + \sigma^2_u + \sigma^2_m \right], \]

where \( \sigma^2_g, \sigma^2_u \) and \( \sigma^2_m \) are the variances of output gap, inflation and money demand, which are the \( \text{VAR}_g, \text{VAR}_u \) and \( \text{VAR}_m \) described in the equation (9). When the MAS is changed by the proportional to \( \hat{\alpha}_1 \), this indicates the existence of \( \epsilon_t \). Then the Fed can adjust its interest rates in response to \( \epsilon_t \) by the amount which is determined by \( \hat{\phi}_m \) from the section V. The Fed can also adjust its interest rates in response to \( \eta_t \) through \( \hat{\alpha}_2 \) as well. As long as the variance of money demand is small, the MAS is a good tool for the Fed to adjust interest rates in response to the unobserved \( \epsilon_t \) and \( \eta_t \). The bigger the variance of money demand, the less effective the MAS will be. When the variance of money demand approaches to infinity, the numerical value of the MAS will be zero. This implies that the MAS may not be used as signal for unobserved \( \epsilon_t \) and \( \eta_t \) when the variance of money demand is big. Figure 4 shows this negative relation between \( \sigma^2_m, \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \).
VII. The Social Welfare Loss and The “Noise” in Money Demand

The variance of money demand $\sigma^2_m$ plays an important role for improving social welfare loss from the MAS when $\varepsilon_t$ and $\eta_t$ are unobserved. In the section V, the smaller the $\sigma^2_m$, the better improvement of the social welfare will be, i.e., $L3 > L4$. In the section VI, the smaller the $\sigma^2_m$, the more effective the MAS will be. When should we stop using the MAS? Figure 5 shows boundary values of $\sigma^2_m$ for using the MAS. In
figure 5, the curve $L5$ shows the positive relation between social welfare loss and $\sigma_m^2$.\(^\text{15}\)

The curve $a1\hat{\text{}}$ shows that the coefficient value on $\varepsilon_t$ decreases when $\sigma_m^2$ increases. The curve $a2\hat{\text{}}$ shows that the coefficient value on $\eta_t$ decreases when $\sigma_m^2$ increases. The intersection between $L5$ and $a1\hat{\text{}}$ is a boundary point where the MAS for $\varepsilon_t$ will not be a good indicator when $\sigma_m^2$ is bigger than the value at intersection. The intersection between $L5$ and $a2\hat{\text{}}$ is another boundary point where the MAS for $\eta_t$ will not be a good indicator when $\sigma_m^2$ is bigger than the value at the intersection. In this case the value of $\sigma_m^2$ at intersection with $a2\hat{\text{}}$ is 1.145 and the value of $\sigma_m^2$ at intersection with $a1\hat{\text{}}$ is 2.472. According Baba, Hendry and Starr’s (1992) survey of demand for money in U.S., the biggest $\sigma_m^2$ is 2.8% from Cooley-Leroy (1981)’s study and the smallest $\sigma_m^2$ is 0.31% from Gordon (1984)’s study.\(^\text{16}\) Thus, practically speaking, the social welfare loss will be improved from the MAS.

---

\(^\text{15}\) The social welfare loss costed by $\sigma_m^2$ is expressed as: $L5 = \frac{1}{2} \left( k^2 + \Gamma \right) \left( \frac{\phi_{\hat{\eta}}}{\sigma^2} \right) \sigma_m^2$. $L5$ is not determined by $\varphi_x$ and $\varphi_{\pi}$.

\(^\text{16}\) See Table 2 in the Appendix D for Baba, Hendry and Starr (1992)’s table of the standard errors of U.S. real M1 demand studies.
VIII. An Example: Measuring Improvement from the MAS

The effectiveness of the MAS is determined by $\phi_m$ and $\sigma_m^2$. And the value of $\phi_m$ is determined by the inverse of the interest rate elasticity $b$. I examine the values of $b$ and $\sigma_m^2$ to $W$ using Walsh (2010)'s parameters which $k = 0.05$, $\beta = 0.99$, $\sigma = 1$, $\rho = 0.5$ and $\Gamma = 0.25$.\footnote{Marest and Thurston (2017) reported the parameter estimations in the literature of NKM. Their report also includes Walsh (2010)'s parameters. See Table 1 in the Appendix D for details.} For the value of $\sigma_m^2$, I use Cooley and Leroy (1981)'s standard error of M1 demand $\sigma_m^2 = 0.028$ which is the largest one according to Baba, Hendry and Starr
The value of $b$ is set to be 2 based on Lucas (2000)’s estimation of 0.5 for the interest rate elasticity of money demand (M1).

First, Figure 6 shows that $b = 2$ as a valid estimation for improving $L$ because $\hat{\phi}_m$ is nonzero positive value at $b = 2$. This figure also implies that the interest rate elasticity of money demand (M1) may play less important role in monetary policy than conventional beliefs. Whether the MSA is measured with a stable money demand to interest rates or not, the social welfare loss should be still improved by the MAS.

Figure 6: The Relationship between $\hat{\phi}_m$ and $b$

Figure 7 shows the improvement of social welfare loss from the MAS. The curve $L3$ is below the curve $L2$ which indicates the improvement of social welfare loss from the MAS. W0 curve is the optimal social welfare loss. $L1$ curve shows the outcome of
social welfare loss when $\varphi_g = 0$ and the existence of g-shocks. Table 3 in the Appendix D compares the improvement of the social welfare loss to different values of $b$ and $\Gamma$.

![Figure 7. The Improvement of $L$ from the MAS](image)

The social welfare loss increases when $\sigma_m^2$ increases and the effectiveness of the MAS decreases when $\sigma_m^2$ increases. Figure 8 shows that the MAS is effective for improving the social welfare loss because the biggest value of the $\sigma_m^2$ is 0.028. For the MAS to be ineffective, $\sigma_m^2$ must be at least greater than 8.11. The a1hat curve is lower than a2hat curve in the Figure 8 but the a1hat curve is higher than a2hat curve in the Figure 5. This is because $b = 2$ in the Figure 8 and $b = 0.5$ in the Figure 5.
IX. Conclusion

The Taylor rule and its coefficients have their limits for improving the social welfare loss when current shocks are unobservable. This paper shows that the social welfare loss is improved from the MAS. $b$ should not affect the effectiveness of the MAS. $\sigma_m^2$ should not affect the validity of the combination policy of the expected standard Taylor rule and the MAS.
References


Appendix A

This appendix shows the Maple commands for obtaining the general solutions of $x_t$ and $\pi_t$ by using the baseline NKM which has the standard Taylor rule. The Maple commands for the general solutions of $x_t$ and $\pi_t$ are as follow:

In this case, the standard Taylor rule (STR) is $i_t = \phi_x x_t + \phi_\pi \pi_t$, which does not have $\phi g$ in it.

Baseline IS curve (BIS) expresses a NK-IS equation:
\[
BIS := x_t = EX \frac{1}{\sigma} \left( i_t - EP \right) + g_t
\]
Note: EX is $E_t[x_{t+1}]$ and EP is $E_t[\pi_{t+1}]$, EX and EP are forward-looking variables. The demand shock, $g_t$, is an AR(1) process $g_t = \lambda g_{t-1} + \epsilon_t$.

Baseline Philips curve (BPC) expresses a NK-PC equation as:
\[
BPC := \pi_t = k \cdot x_t + \beta \cdot EP + u_t
\]
Note: The cost shock, $u_t$, is an AR(1) process $u_t = \rho u_{t-1} + \eta_t$.

The standard Taylor Rule (STR) expresses a Taylor rule using actual values of x and $\pi$.
\[
STR := i_t = \phi_x x_t + \phi_\pi \pi_t
\]

The undetermined coefficients equations of x and $\pi$ are represented as (4) and (5) below
\[
BIS1 := x_t = a_1 \cdot g_t + a_2 \cdot u_t
\]
\[
BPC1 := \pi_t = b_1 \cdot g_t + b_2 \cdot u_t
\]

Obtaining the general solutions of x and $\pi$ by solving (4) and (5) using the Baseline NKM (1) and (2), and the cTR (3):

Equation (6) below shows x is updated one period ahead:
\[
BIS2 := x_{t+1} = a_1 \cdot g_{t+1} + a_2 \cdot u_{t+1}
\]

Equation (7) below shows $\pi$ is updated one period ahead:
\[
BPC2 := \pi_{t+1} = b_1 \cdot g_{t+1} + b_2 \cdot u_{t+1}
\]

Equation (8) below shows $g[t+1]$ and $u[t+1]$ in (6) are replaced by $\lambda g[t]+\epsilon[t+1]$ and $\rho u[t]+\eta[t+1]:$
\[
BIS3 := subs( g_{t+1} = \lambda \cdot g_t + \epsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BIS2 )
\]

Equation (9) below shows $g[t+1]$ and $u[t+1]$ in (7) are replaced by $\lambda g[t]+\epsilon[t+1]$ and $\rho u[t]+\eta[t+1]:$
\[
BPC3 := subs( g_{t+1} = \lambda \cdot g_t + \epsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BPC2 )
\]

Equation (10) below shows the undetermined coefficient equation of EX in the Baseline NKM:
\[
EBIS3 := EX = a_1 \lambda \cdot g_t + a_2 \cdot \rho \cdot u_t
\]
Note: The expected values of $\varepsilon_{t+1}$ and $\eta_{t+1}$ are equal to zero.

Equation (11) below shows the undetermined coefficient equation of EP in the Baseline NKM:

$$EBPC3 := EP = b1 \cdot \lambda \cdot g_t + b2 \cdot \rho \cdot u_t$$

Note: The expected values of $\varepsilon_{t+1}$ and $\eta_{t+1}$ are equal to zero.

Equation (12) below shows the $x$ and $\pi$ in STR (3) are replaced by (6) and (7):

$$STR1 := subs(BIS1, BPC1, STR)$$

Equation (13) below shows $x$, $i$, EX and EP in Baseline IS (1) are replaced by (6), (12), (10) and (11):

$$soli1 := subs(BIS1, STR1, BIS3, EBPC3, BIS)$$

Equation (14) below shows $x$, $\pi$ and EP in Baseline Philips curve (2) are replaced by (6), (7) and (11):

$$sol02 := subs(BIS1, BPC1, EBPC3, BPC)$$

Then I am able to solve (13) and (14) by the method of undetermined coefficient:

$$con1 := subs(g_t = 0, u_t = 0, soli1)$$
$$con2 := subs(g_t = 1, u_t = 0, soli1)$$
$$con3 := subs(g_t = 0, u_t = 1, soli1)$$
$$con4 := subs(g_t = 0, u_t = 0, sol02)$$
$$con5 := subs(g_t = 1, u_t = 0, sol02)$$
$$con6 := subs(g_t = 0, u_t = 1, sol02)$$

$$OP := solve\{\{con2, con3, con5, con6\}, \{a1, a2, b1, b2\}\}$$

Below is the general solution of $x$ when the standard TR does not have $\phi[g]$ in it.

$$solx := subs(OP, BIS1)$$

Below is the general solution of $\pi$ when the standard TR does not have $\phi[g]$ in it.

$$solp := subs(OP, BPC1)$$

The outcomes of the Maple commands are the general solutions which

$$x_t = \frac{\rho - \phi_\pi}{\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi_\pi - \rho \sigma + \sigma + \phi_x} u_t - \frac{\sigma (\beta \lambda - 1)}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \lambda \phi_x - k \lambda + k \phi_\pi - \lambda \sigma + \sigma + \phi_x} g_t$$

and
\[ \pi_t = -\frac{\rho \sigma - \sigma - \varphi_x}{\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \varphi_x - k \rho + k \varphi_{\pi} - \rho \sigma + \sigma + \varphi_x} u_t + \frac{\sigma k}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \lambda \varphi_x - k \lambda + k \varphi_{\pi} - \lambda \sigma + \sigma + \varphi_x} g_t. \]

**Appendix B**

The constrained solution of the minimum social welfare loss requires two conditions regarding the f.o.c. which is derived from the general solutions based on the Appendix A. The f.o.c. w.r.t. \( \varphi_{\pi} \) is

\[
\frac{1}{2} \left[ -\frac{2 \Gamma (\rho - \varphi_{\pi})}{(\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \varphi_x - k \rho + k \varphi_{\pi} - \rho \sigma + \sigma + \varphi_x)^2} \right] \frac{1}{2} \left[ VArU + \frac{1}{2} \left( \begin{array}{c}
\frac{2 \Gamma (\rho - \varphi_x)}{2 \Gamma (\rho - \varphi_x)^2 k} \\
\frac{2 \Gamma (\beta \lambda - 1)^2 \sigma^2 k}{(\beta \lambda^2 \sigma - \beta \lambda \varphi_x - k \lambda + k \varphi_{\pi} - \lambda \sigma + \sigma + \varphi_x)^3} \\
\frac{2 k^2 \sigma^2}{(\beta \lambda^2 \sigma - \beta \lambda \varphi_x - k \lambda + k \varphi_{\pi} - \lambda \sigma + \sigma + \varphi_x)^3}
\end{array} \right) VArG \right] \]

The first condition is that the denominators of the f.o.c. must be non-zero values so \( \varphi_{\pi} \neq \rho \) and \( \varphi_x \neq \sigma (\rho - 1) \) in the bracket of \( VArU \) and \( \varphi_{\pi} \neq \lambda \) and \( \varphi_x \neq \sigma (\lambda - 1) \) in the bracket of \( VArG \). In other words, if
\[
\frac{\varphi_x}{\varphi_\pi - 1} = \sigma
\]

the overall value of the f.o.c. is undefined because of the problem of division by zero.\(^\text{18}\)

The second condition is regarding the numerators of the f.o.c. whose value must be zero for the minimum social welfare loss. When

\[
\Gamma = -\frac{k^2}{(\beta \lambda - 1)^2},
\]

the second term in the right-hand side of the f.o.c. is zero. When

\[
\varphi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k \sigma (\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho
\]

or

\[
\varphi_\pi = \rho,
\]

the first term in the right-hand side of the f.o.c. is zero. The first part of this condition is the same as Thurston (2010/2012)’s constraint on \(\varphi_x\) and \(\varphi_\pi\).

We then have two possible constrained solutions of the social welfare loss. When

\[
\frac{\varphi_x}{\varphi_\pi - 1} \neq \sigma \quad \text{and} \quad \Gamma = -\frac{k^2}{(\beta \lambda - 1)^2}, \quad \text{if} \quad \varphi_\pi = \rho,
\]

the constrained solution is

\[
\frac{1}{(\beta \rho - 1)^2} \text{VARu},
\]

which is bigger than the unconstrained solution \(\frac{\Gamma}{(\beta \rho - 1)^2 \Gamma + k^2 \text{VARu}}, \) whose Taylor rule has \(\varphi_g = \sigma\). When \(\frac{\varphi_x}{\varphi_\pi - 1} \neq \sigma \quad \text{and} \quad \Gamma = -\frac{k^2}{(\beta \lambda - 1)^2}, \quad \text{if}\)

\[
\frac{\varphi_\pi - \varphi_\pi}{\varphi_\pi - 1} = \rho \quad \text{and} \quad \Gamma = -\frac{k^2}{(\beta \lambda - 1)^2}, \quad \text{if} \quad \varphi_\pi = \lambda.
\]

\(^\text{18}\) To get this condition, first let \(\varphi_\pi = \rho\) and \(\varphi_x = \sigma (\rho - 1)\). Next we replace \(\rho\) by \(\varphi_\pi\) in the right-hand side of \(\varphi_x\) and rearrange terms to obtain the condition: \(\frac{\varphi_x}{\varphi_\pi - 1} = \sigma\). We also get the same condition if we use \(\varphi_\pi = \lambda\) and \(\varphi_x = \sigma (\lambda - 1)\).
Unobservable Shocks and the Money as Signal

Tzu-Hao Huang

05/05/2017

\[ \varphi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k \sigma (\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho, \]  
the constrained solution is

\[ \frac{1}{(\beta \rho + \beta \lambda - 2)(\rho - \lambda)} VARu, \]

which is again bigger than the unconstrained solution.

Appendix C

Figure C. shows the value of \( \varphi_m \) when \( b \) is approaching positive (negative) infinity with \( k = 0.3, \beta = 0.99 \) and \( \sigma = 1 \).

![Figure C. The Value of \( \hat{\varphi}_m \)](image)
Appendix D

I reproduce Marest and Thurston (2017)’s table 3 here and I give a comment on the use of $\Gamma$ in the Note.

Table 1
Parameters Used in Literature

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Parameters</th>
<th>$k$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\Gamma$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Woodford</td>
<td>2003</td>
<td></td>
<td>0.024</td>
<td>0.99</td>
<td>0.16</td>
<td>0.003</td>
<td>0.4</td>
</tr>
<tr>
<td>Billi</td>
<td>2008</td>
<td></td>
<td>0.024</td>
<td>0.9926</td>
<td>0.16</td>
<td>0.003</td>
<td>0.1</td>
</tr>
<tr>
<td>Walsh</td>
<td>2010</td>
<td></td>
<td>0.05</td>
<td>0.99</td>
<td>1</td>
<td>0.25</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Note: We should ignore the small values of $\Gamma$ when we compare the improvement of $L$. The smaller the $\Gamma$, the smaller the $L$ will be. Small $\Gamma$ implies that the weight on output gap is small and $L$ is less affected by the changes in output gap from $g_t$. Since the MAS improved $L$ mainly by improving output gap, an almost zero weight on $x$ send a wrong message that the MAS was ineffective.

I reproduce Baba, Hendry and Starr (1992) table 2 here.

Table 2
$\sigma^2_m$ used in Literature

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Standard Error of M1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldfeld</td>
<td>1973</td>
<td>0.43%</td>
</tr>
<tr>
<td>Garcia and Pak</td>
<td>1979</td>
<td>0.63%</td>
</tr>
<tr>
<td>Rose</td>
<td>1985</td>
<td>0.48%</td>
</tr>
<tr>
<td>Gordon</td>
<td>1984</td>
<td>0.43%</td>
</tr>
<tr>
<td>McAleer, Pagan and Volker</td>
<td>1985</td>
<td>0.31%</td>
</tr>
<tr>
<td>Simpson and Porter</td>
<td>1980</td>
<td>0.52% - 0.59%</td>
</tr>
<tr>
<td>Cooley and Leroy</td>
<td>1981</td>
<td>2.80%</td>
</tr>
<tr>
<td>Baba, Hendry and Starr</td>
<td>1992</td>
<td>0.38%</td>
</tr>
</tbody>
</table>

Here I show the relationship between $b$, $\Gamma$ and The Improvement of $L$.

Table 3
The Relationship between $b$, $\Gamma$ and $L$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$\Gamma$</th>
<th>The Improvement of $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>big ($\Gamma &gt; b$)</td>
<td>big</td>
</tr>
<tr>
<td>big</td>
<td>big ($\Gamma &gt; b$)</td>
<td>small</td>
</tr>
<tr>
<td>small</td>
<td>small</td>
<td>almost no improvement</td>
</tr>
</tbody>
</table>