

# Underborrowing\*

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## Abstract

This paper establishes the existence of underborrowing in an infinite-horizon open-economy model with financial frictions in which the value of tradable and nontradable endowments serves as collateral. Underborrowing emerges in the present context because this economy is prone to self-fulfilling financial crises. In this environment individual agents engage in excessive precautionary savings as a way to self-insure. This finding stands in contrast to the overborrowing result stressed in the related literature and is shown to obtain for plausible calibrations.

**JEL classification:** E44, F41, G01, H23.

**Keywords:** Pecuniary externalities, collateral constraints, overborrowing, underborrowing, financial crises, capital controls.

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# 1 Introduction

Many open-economy models with collateral constraints display a pecuniary externality originating in the fact that the price of objects pledgable as collateral is taken as given by individual agents but is endogenous in equilibrium. A result stressed in the literature is that these economies overborrow, that is, they borrow more than they would if agents internalized the externality (Auernheimer and García-Saltos, 2000; Bianchi, 2011; Korinek, 2011; Jeanne and Korinek, 2010).

This paper establishes the existence of underborrowing in a workhorse version of this class of models, that is, an infinite-horizon open-economy with financial frictions in which the value of tradable and nontradable income serves as collateral. These results, which stand in contrast to the overborrowing result stressed in the related literature, emerge because the economy is prone to self-fulfilling financial crises. In this environment individual agents engage in excessive precautionary saving as a way to self-insure.

The underborrowing result presented in this paper stems from the fact that the collateral constraint model can generate non-convexities in the equilibrium resource constraint. Specifically, although the collateral constraint is well behaved at the individual level, in the sense that it tightens when individuals borrow more, it may be ill-behaved at the aggregate level, in the sense that it may relax as aggregate borrowing increases. Such a perverse relationship can give rise to multiple equilibria, as suggested heuristically by Jeanne and Korinek (2010) in the context of an economy with a stock collateral constraint and by Mendoza (2005) in the context of an economy with a flow collateral constraint.

This paper formally establishes that collateral constraints can give rise to multiple equilibria in the context of open economy models with flow collateral constraints. We focus on flow collateral constraints in which tradable and nontradable output have collateral value, which is the type of flow collateral constraint most frequently studied in the related literature. Under this formulation, the source of pecuniary externalities is the relative price of nontradable goods in terms of tradables, or the real exchange rate. The collateral constraint gives rise to a pecuniary externality because of two features of the model: First, individual households fail to internalize the effect of their borrowing decision on the relative price of nontradables and hence the value of their own collateral; and second, the relative price of nontradables enters in the social planner's constraints. We show that self-fulfilling financial crises can emerge as a result of pessimistic views about the value of collateral that induce agents to deleverage. The underborrowing result derived in this paper is of interest because the type of collateral constraint we study is widely used in the quantitative open-economy literature (e.g., Bianchi, 2011; Benigno et al. 2013 and 2014; Ottonello, 2015).

The second contribution of the paper is to explicitly address the issue of implementation. As is well known, optimal policy is mute in this regard. In the context of the present analysis, this applies to optimal capital-control policy. The capital control policy that is consistent with the optimal allocation can also be consistent with other (non-optimal) allocations. A natural question is therefore what kind of capital control policy can implement the optimal allocation. We show that capital control policies that are triggered by sudden and discrete bursts in capital outflows can avoid self-fulfilling financial crises and implement the optimal allocation. According to this class of capital control policies, the government threatens to tax capital flight if a panic attack induces agents to collectively deleverage. This threat discourages nonfundamental runs on the country's debt, leaving as the sole possible equilibrium the optimal one.

The third contribution of the paper is quantitative. Existing quantitative studies avoid the multiplicity problem by choosing calibrations for which nonconvexities are absent. This concern in choosing model parameterizations is explicitly mentioned, for instance, in Jeanne and Korinek (2010) in the context of a stock-collateral-constraint model and in Benigno et al. (2014) in the context of a flow-collateral-constraint model, and is implicit in the parameterizations adopted in Bianchi (2011) and Ottonello (2015), among others. The present paper solves for equilibrium dynamics in the presence of nonconvexities. We show that under plausible calibrations, the presence of nonconvexities can give rise to equilibria exhibiting underborrowing. This result stands in contrast to the overborrowing result stressed in the related quantitative literature.

In an economy calibrated with parameters typically used in the emerging-market business-cycle literature and fed with shocks estimated on quarterly Argentine data, we find equilibria in which the unregulated economy underborrows. A byproduct of the analytical analysis is a diagnostic test that is readily applicable and can be of use to quantitative researchers seeking to ascertain whether their parameterizations give rise to multiplicity of equilibria.

This paper is related to several branches of the literature on credit frictions in macroeconomics. The type of flow collateral constraint we study was introduced in open economy models by Mendoza (2002) to understand sudden stops caused by fundamental shocks. The externality that emerges when debt is denominated in tradables goods but partly leveraged on nontradable income and the consequent room for macroprudential policy was emphasized by Korinek (2007) in the context of a three-period model. Bianchi (2011) shows, in the context of an infinite horizon model, that the externality can lead to overborrowing and characterizes the behavior of optimal prudential policy. An exception to the standard overborrowing result is Benigno et al. (2013). However, the cause of underborrowing in the Benigno et al. model is of a different nature from the one identified in the present paper. In

their work underborrowing stems from introducing production in the nontradable sector or distortionary taxation. The result of the Benigno et al. paper is complementary but different from the one presented here. In the present study, underborrowing arises even in the context of an endowment economy and without distortionary taxation and is due to the fragility created by the possibility created by self-fulfilling crises.

The remainder of the paper is organized as follows. Section 2 presents an open economy with a flow collateral constraint in which tradable and nontradable output have collateral value. Section 3 characterizes steady-state equilibria. Section 4 characterizes analytically multiplicity of equilibrium. It shows the existence of up to two equilibria with self-fulfilling crashes in the value of collateral. Section 5 introduces nonfundamental uncertainty (sunspots) and shows that it can give rise to persistent self-fulfilling financial crises. Section 6 studies optimal capital control policy. It shows that the unregulated economy underborrows relative to the economy with optimal capital controls. Section 7 presents a capital-control policy rule that can implement the optimal allocation. Section 8 quantitatively characterizes debt dynamics in a stochastic economy with output and interest-rate shocks in which agents coordinate on equilibria driven by pessimistic beliefs and establishes that underborrowing occurs under plausible calibrations. Section 9 concludes.

## 2 The Model

Consider a small open endowment economy in which households have preferences of the form

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad (1)$$

where  $c_t$  denotes consumption in period  $t$ ,  $U(\cdot)$  denotes an increasing and concave period utility function,  $\beta \in (0, 1)$  denotes a subjective discount factor, and  $\mathbb{E}_t$  denotes the expectations operator conditional on information available in period  $t$ . The period utility function takes the CRRA form  $U(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$  with  $\sigma > 0$ . We assume that consumption is a composite of tradable and nontradable goods, taking the CES form

$$c_t = A(c_t^T, c_t^N) \equiv \left[ a c_t^{T^{1-1/\xi}} + (1-a) c_t^{N^{1-1/\xi}} \right]^{1/(1-1/\xi)}, \quad (2)$$

with  $\xi > 0$ ,  $a \in (0, 1)$ , and where  $c_t^T$  denotes consumption of tradables in period  $t$  and  $c_t^N$  denotes consumption of nontradables in period  $t$ . Households are assumed to have access to a single, one-period, risk-free, internationally-traded bond denominated in terms of tradable goods that pays the interest rate  $r_t$  when held from period  $t$  to period  $t+1$ . The household's

sequential budget constraint is given by

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{1 + r_t}, \quad (3)$$

where  $d_t$  denotes the amount of debt due in period  $t$  and  $d_{t+1}$  denotes the amount of debt assumed in period  $t$  and maturing in  $t + 1$ . The variable  $p_t$  denotes the relative price of nontradables in terms of tradables, and  $y_t^T$  and  $y_t^N$  denote the endowments of tradables and nontradables, respectively. Both endowments are assumed to be exogenously given. The collateral constraint takes the form

$$d_{t+1} \leq \kappa(y_t^T + p_t y_t^N), \quad (4)$$

where  $\kappa > 0$  is a parameter. Throughout this paper, we will assume that  $\kappa < (1 + r)/r$ , where  $r$  is the steady-state real interest rate. This assumption makes the collateral constraint nontrivial, in the sense that higher values of  $\kappa$  would imply that the collateral constraint is slack even at the natural debt limit. This restriction is also empirically reasonable. Suppose that the interest rate is 5 percent in annual terms. Then the upper bound of debt is 21 annual outputs.

The borrowing constraint introduces an externality, because each individual household takes the real exchange rate,  $p_t$ , as exogenously determined, even though their collective absorptions of nontradable goods are a key determinant of this relative price. From the perspective of the individual household, the collateral constraint is well behaved in the sense that the higher the debt level is, the tighter the collateral constraint will be. As we shall see shortly, this may not be the case in equilibrium.

Households choose a set of processes  $\{c_t^T, c_t^N, c_t, d_{t+1}\}$  to maximize (1) subject to (2)-(4), given the processes  $\{r_t, p_t, y_t^T, y_t^N\}$  and the initial debt position  $d_0$ . The first-order conditions of this problem are (2)-(4) and

$$U'(A(c_t^T, c_t^N))A_1(c_t^T, c_t^N) = \lambda_t, \quad (5)$$

$$p_t = \frac{1 - a}{a} \left( \frac{c_t^T}{c_t^N} \right)^{1/\xi}, \quad (6)$$

$$\left( \frac{1}{1 + r_t} - \mu_t \right) \lambda_t = \beta \mathbb{E}_t \lambda_{t+1}, \quad (7)$$

$$\mu_t \geq 0, \quad (8)$$

and

$$\mu_t [d_{t+1} - \kappa(y_t^T + p_t y_t^N)] = 0, \quad (9)$$

where  $\beta^t \lambda_t$  and  $\beta^t \lambda_t \mu_t$  denote the Lagrange multipliers on the sequential budget constraint (3) and the collateral constraint (4), respectively. As usual, the Euler equation (7) equates the marginal benefit of assuming more debt with its marginal cost. During tranquil times, when the collateral constraint does not bind, one unit of debt payable in  $t + 1$  increases tradable consumption by  $1/(1 + r_t)$  units in period  $t$ , which increases utility by  $\lambda_t/(1 + r_t)$ . Thus, in tranquil times the marginal benefit of debt is  $\lambda_t/(1 + r_t)$ . The marginal cost of debt assumed in period  $t$  and payable in  $t + 1$  is the marginal utility of consumption in period  $t + 1$  discounted at the subjective discount factor,  $\beta \mathbb{E}_t \lambda_{t+1}$ . During financial crises, when the collateral constraint binds, the marginal utility of debt falls  $\lambda_t/(1 + r_t)$  to  $[1/(1 + r_t) - \mu_t] \lambda_t$ , reflecting a shadow penalty for trying to increase debt when the collateral constraint is binding.

In equilibrium, the market for nontradables must clear. That is,

$$c_t^N = y_t^N.$$

Then, using this expression and equations (5) and (6) to eliminate  $c_t^N$ ,  $\lambda_t$ , and  $p_t$ , from the household's first-order conditions, we can define a competitive equilibrium as a set of processes  $\{c_t^T, d_{t+1}, \mu_t\}$  satisfying

$$\left( \frac{1}{1 + r_t} - \mu_t \right) U'(A(c_t^T, y_t^N)) A_1(c_t^T, y_t^N) = \beta \mathbb{E}_t U'(A(c_{t+1}^T, y_{t+1}^N)) A_1(c_{t+1}^T, y_{t+1}^N), \quad (10)$$

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1 + r_t}, \quad (11)$$

$$d_{t+1} \leq \kappa \left[ y_t^T + \left( \frac{1 - a}{a} \right) c_t^{T^{1/\xi}} y_t^{N^{1-1/\xi}} \right], \quad (12)$$

$$\mu_t \left[ \kappa y_t^T + \kappa \left( \frac{1 - a}{a} \right) c_t^{T^{1/\xi}} y_t^{N^{1-1/\xi}} - d_{t+1} \right] = 0, \quad (13)$$

$$\mu_t \geq 0, \quad (14)$$

given the exogenous processes  $\{r_t, y_t^T, y_t^N\}$  and the initial condition  $d_0$ .

The fact that  $c_t^T$  appears on the right-hand side of the equilibrium version of the collateral constraint, equilibrium condition (12), means that during contractions in which the absorption of tradables falls the collateral constraint endogenously tightens. Individual agents do not take this effect into account in choosing their consumption plans. This is the nature of

the pecuniary externality in this model.

As we saw earlier, the individual collateral constraint is well behaved in the sense that it tightens as the level of debt increases. This may not be the case at the aggregate level. To see this, use equilibrium condition (11) to eliminate  $c_t^T$  from equilibrium condition (12) to obtain

$$d_{t+1} \leq \kappa \left[ y_t^T + \left( \frac{1-a}{a} \right) \left( y_t^T + \frac{d_{t+1}}{1+r_t} - d_t \right)^{1/\xi} y_t^{N^{1-1/\xi}} \right].$$

It is clear from this expression that the right-hand side is increasing in the equilibrium level of external debt,  $d_{t+1}$ . Moreover, depending on the values assumed by the parameters  $\kappa$ ,  $a$ , and  $\xi$ , the right-hand side may increase more than one for one with  $d_{t+1}$ . In this case an increase in debt, instead of tightening the collateral constraint may relax it. In other words, the more indebted the economy becomes, the less leveraged it will be. As we will see shortly, this possibility can give rise to multiple equilibria and self-fulfilling drops in the value of collateral.

Furthermore, while the individual household's constraints represent a convex set, the equilibrium aggregate resource constraint may not. To see this examine first the restrictions faced by the individual household. If two debt levels  $d^1$  and  $d^2$  satisfy (3) and (4), then any weighted average  $\alpha d^1 + (1-\alpha)d^2$  for  $\alpha \in [0,1]$  also satisfies these two conditions. From an equilibrium perspective, however, this ceases to be true in general. If the intratemporal elasticity of substitution  $\xi$  is less than unity, which is the case of greatest empirical relevance for many countries (Akinci, 2011), the equilibrium value of collateral is convex in the level of debt. This property may cause the emergence of two distinct values of  $d_{t+1}$  for which the collateral constraint binds and two disjoint intervals of debt levels for which the collateral constraint is slack, rendering the feasible set of debts nonconvex.

The focus of sections 3 through 7 is to analytically characterize conditions for the existence of self-fulfilling financial crises and the design and implementation of optimal capital-control policy in the present model. For analytical convenience, in those sections we impose the following assumptions: The tradable and nontradable endowments and the interest rate are constant and equal to  $y_t^T = y^T$ ,  $y_t^N = 1$ , and  $r_t = r$ , for all  $t$ , respectively. Finally, we set  $\beta(1+r) = 1$ . Given these assumptions, the equilibrium conditions (10)-(13) can be written as

$$\Lambda(c_t^T) [1 - (1+r)\mu_t] = \Lambda(c_{t+1}^T), \quad (15)$$

$$c_t^T + d_t = y^T + \frac{d_{t+1}}{1+r}, \quad (16)$$

$$d_{t+1} \leq \kappa \left[ y^T + \frac{(1-a)}{a} \left( y^T + \frac{d_{t+1}}{(1+r)} - d_t \right)^{\frac{1}{\xi}} \right], \quad (17)$$

$$\mu_t \left\{ \kappa \left[ y^T + \frac{(1-a)}{a} \left( y^T + \frac{d_{t+1}}{(1+r)} - d_t \right)^{\frac{1}{\xi}} \right] - d_{t+1} \right\} = 0, \quad (18)$$

$$\mu_t \geq 0, \quad (19)$$

and

$$c_t^T > 0, \quad (20)$$

with  $d_0$  given, where

$$\Lambda(c_t^T) \equiv U'(A(c_t^T, 1))A_1(c_t^T, 1),$$

denotes the equilibrium level of the marginal utility of tradable consumption. Given the assumed concavity of  $U(\cdot)$  and  $A(\cdot, \cdot)$ ,  $\Lambda(\cdot)$  is a decreasing function.

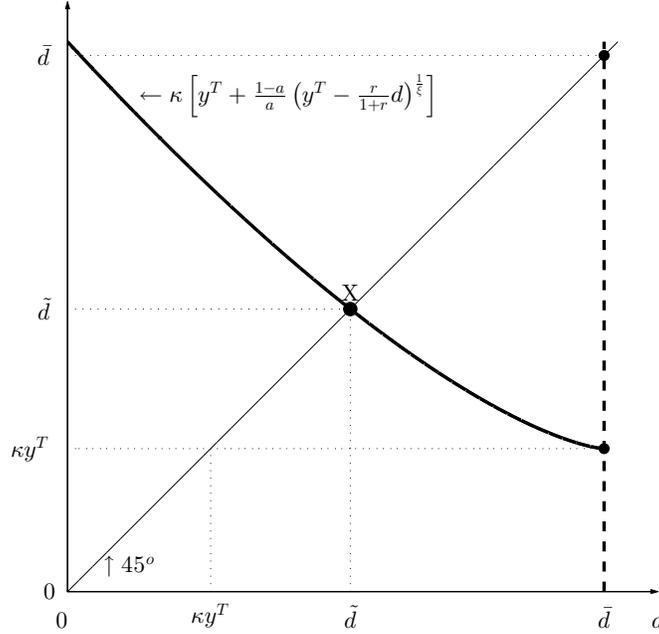
### 3 Steady-State Equilibria

We first characterize conditions for the existence of an equilibrium in which traded consumption and debt are constant for all  $t \geq 0$ , that is, an equilibrium in which  $c_t^T = c_0^T$  and  $d_t = d_0$  for all  $t \geq 0$ , where  $d_0$  is a given initial condition. We refer to this equilibrium as a steady-state equilibrium. By (15), in a steady-state equilibrium  $\mu_t = 0$  for all  $t$ . This means that in a steady-state equilibrium the slackness condition (18) and the nonnegativity constraint (19) are also satisfied for all  $t$ . When  $d_{t+1} = d_t = d$ , the collateral constraint (17) becomes

$$d \leq \kappa \left[ y^T + \frac{(1-a)}{a} \left( y^T - \frac{r}{(1+r)}d \right)^{\frac{1}{\xi}} \right]. \quad (21)$$

We refer to this expression as the steady-state collateral constraint. Figure 1 displays the left- and right-hand sides of the steady-state collateral constraint as a function of  $d$ . The left-hand side is the 45-degree line. The right-hand side, shown with a thick solid line, is the steady-state value of collateral. By (16), steady-state consumption of tradables is given by  $c^T = y^T - \frac{r}{1+r}d$ . By equilibrium condition (20),  $c^T$  must be positive. Let  $\bar{d} \equiv y^T(1+r)/r$  denote the natural debt limit, defined as the highest level of debt consistent with a nonnegative constant stream of tradable consumption. At the natural debt limit,  $c^T = 0$ , for  $d$  below the natural debt limit,  $c^T > 0$ , and for  $d > \bar{d}$ ,  $c^T < 0$ . This means that a steady-state equilibrium can only exist for  $d < \bar{d}$ . For values of debt between zero and  $\bar{d}$  the right-hand side of (21) is downward sloping. (Recall that  $\xi > 0$ .) It follows that the steady-state collateral constraint is well behaved in the sense that the higher the steady-state level of debt is, the tighter the steady-state collateral constraint will be. The left- and right-hand sides of (21) intersect once somewhere in the interval  $[0, \bar{d}]$ . To see this, note first that the left-hand side of the

Figure 1: Feasible Debt Levels in the Steady State



steady-state collateral constraint is upward sloping while the right-hand side is downward sloping. At  $d = \bar{d}$ , the right-hand side of the steady-state collateral constraint equals  $\kappa y^T$  and at  $d = 0$  it equals  $\kappa y^T + \kappa(1-a)/a y^{T1/\xi} > 0$ . The left-hand side is  $y^T(1+r)/r$  at  $d = \bar{d}$  and 0 at  $d = 0$ . By the assumption that  $\kappa < (1+r)/r$ , at  $d = \bar{d}$  the left-hand side of (21) is larger than the right-hand side, and at  $d = 0$  the left-hand is smaller than the right-hand side.

Let  $\tilde{d} < \bar{d}$  be the value of  $d$  at which the steady-state collateral constraint (21) holds with equality, that is, the value of  $d$  at which the right-hand side of the steady-state collateral constraint crosses the 45-degree line, point  $X$  in figure 1. Formally,  $\tilde{d}$  is implicitly given by

$$\tilde{d} = \kappa \left[ y^T + \frac{1-a}{a} \left( y^T - \frac{r}{1+r} \tilde{d} \right)^{\frac{1}{\xi}} \right]. \quad (22)$$

Any value of initial debt,  $d_0$ , less than or equal to  $\tilde{d}$  satisfies the steady-state collateral constraint (21). Since we have already shown that a constant value of debt also satisfies all other equilibrium conditions, we have demonstrated that any initial value of debt less than or equal to  $\tilde{d}$  can be supported as a steady-state equilibrium.

## 4 Self-Fulfilling Financial Crises

Do there exist equilibria other than the steady-state equilibrium? The answer turns out to be yes. To show this we characterize conditions under which a second equilibrium exists with the property that the collateral constraint binds in period 0. Recall that in the steady-state equilibrium the collateral constraint is slack. Specifically, under the second equilibrium we want to characterize, in period 0, for non-fundamental reasons agents wake up feeling pessimistic and decide to cut consumption, increase savings, and deleverage. In turn, the contraction in consumption brings down the relative price of nontradables, causing the value of collateral to drop and the collateral constraint to bind, validating agents' pessimistic sentiments. Because of these characteristics, we refer to this second equilibrium as a self-fulfilling financial-crisis equilibrium.

As shown in section 3, in order for the steady-state equilibrium to exist, the initial level of debt,  $d_0$ , must be less than or equal to  $\tilde{d}$ . Thus, we wish to know whether the type of self-fulfilling crisis we just described occurs for initial values of debt less than or equal to  $\tilde{d}$ . In the present analysis, we focus on self-fulfilling crises in which the economy reaches a steady state in period 1:

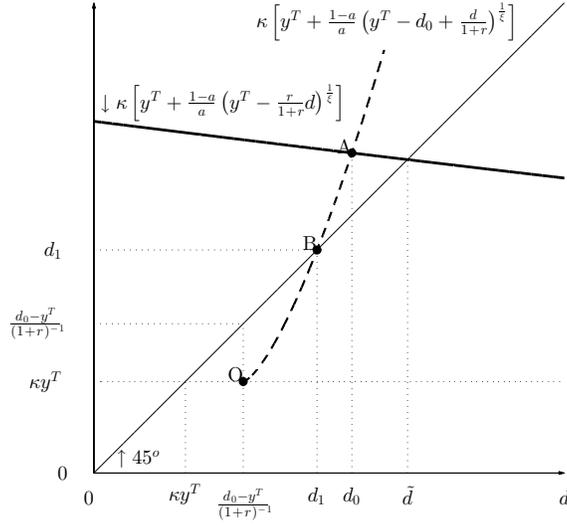
**Definition 1 (Self-Fulfilling Financial-Crisis Equilibrium)** *For any initial level of debt  $d_0 < \tilde{d}$ , a self-fulfilling financial-crisis equilibrium is a set of deterministic paths  $\{c_t^T, d_{t+1}, \mu_t\}_{t=0}^\infty$  satisfying conditions (15)-(20),  $d_1 < d_0$  (deleveraging),  $d_{t+1} = d_1$  (steady state after period 0), where  $\tilde{d}$  is defined in equation (22).*

Consider the collateral constraint in period 0, which is given by

$$d \leq \kappa \left[ y^T + \left( \frac{1-a}{a} \right) \left( y^T + \frac{d}{1+r} - d_0 \right)^{\frac{1}{\xi}} \right], \quad (23)$$

expressed as a function of the level of debt in period 1, denoted by  $d$ . We refer to (23) as the period-0 collateral constraint. Suppose that  $d_0 < \tilde{d}$ , so that a steady-state equilibrium exists. The right-hand side of the period-0 collateral constraint is increasing in  $d$ , which says that the value of collateral is increasing in debt. This is so because more borrowing allows for higher consumption, which in equilibrium leads to an increase in the relative price of nontradables as the supply of nontradables is fixed. Clearly, the right-hand sides of the period-0 and the steady-state collateral constraints intersect when  $d = d_0$ . Thus, since the steady-state collateral constraint is slack at  $d_0$ , so is the period-0 collateral constraint.

Figure 2: Multiple Equilibria: Self-fulfilling Financial Crisis



Note. The figure is drawn under the assumption that  $0 < \xi < 1$ .

#### 4.1 Low Intratemporal Elasticity of Substitution: $0 < \xi < 1$

Suppose  $0 < \xi < 1$ , which, as mentioned earlier, is the case of greatest empirical interest. In subsection 4.3, we consider the case  $\xi > 1$ . When  $\xi \in (0, 1)$ , the period-0 collateral constraint is convex in  $d$ . Figure 2 plots, with a broken line, the right-hand side of the period-0 collateral constraint as a function of the period-1 level of debt chosen in period 0,  $d$ . It also reproduces from figure 1 the right-hand side of the steady-state collateral constraint, the thick solid downward-sloping line. Point  $A$  in the figure is the steady-state equilibrium. If the economy stays forever at point  $A$ , the collateral constraint is always slack, and debt is constant and equal to  $d_0$  at all times.

We now show that point  $B$  in the figure is a self-fulfilling financial-crisis equilibrium. To establish this result, we must show that equilibrium conditions (15)-(20) are satisfied and  $d_{t+1} = d_1 < d_0$ , for all  $t \geq 0$ . In period 0,  $c_0^T$  is positive. To see this, note that because at point  $B$  the right-hand side of the period-0 collateral constraint cuts the 45-degree line from below, its slope must be larger than unity. Let  $S(d; d_0)$  denote the slope of the right-hand side of the period-0 collateral constraint as a function of  $d$  for a given value of  $d_0$ . Then we have that

$$S(d; d_0) \equiv \kappa \left( \frac{1-a}{a} \right) \frac{1}{(1+r)} \frac{1}{\xi} \left( y^T + \frac{d}{1+r} - d_0 \right)^{\frac{1}{\xi}-1}. \quad (24)$$

Note that  $c_0^T = y^T + \frac{d}{1+r} - d_0$ . Thus, the fact that  $S(d; d_0)$  is greater than one at point  $B$

guarantees that  $c_0^T > 0$ , so that equilibrium condition (20) is satisfied in period 0. Because at point  $B$  the collateral constraint is binding in period 0, equilibrium conditions (17) and (18) are satisfied in that period. Also, the facts that in the proposed equilibrium  $d_1 < d_0$  and  $d_1 = d_2$  imply that  $c_0^T < c_1^T$ , which can be verified by comparing the resource constraint (16) evaluated at  $t = 0$  and  $t = 1$ . In turn,  $c_0^T < c_1^T$  implies, by the Euler equation (15), that a strictly positive value of the Lagrange multiplier  $\mu_0$  makes the Euler equation hold with equality in period 0. So equation (19) holds in period 0. This establishes that the debt level associated with point  $B$  satisfies all equilibrium conditions in period 0. Since  $d_1 < \tilde{d}$ , we have, from the analysis of steady-state equilibria in section 3, that  $d_t = d_1$  for all  $t \geq 1$  can be supported as a steady-state equilibrium.

The self-fulfilling financial-crisis equilibrium takes place at a level of period-1 debt at which, from an aggregate point of view, the period-0 collateral constraint behaves perversely in the sense that less borrowing (i.e., more deleveraging) tightens rather than slackens the collateral constraint. Graphically this property is reflected in the fact that at point  $B$  in figure 2 the slope of the right-hand side of the period-0 collateral constraint is greater than one, which means that reducing debt by one unit lowers the value of collateral by more than one unit so that by deleveraging the economy would violate the collateral constraint.

The characterization of self-fulfilling financial-crisis equilibria represented by point  $B$  in figure 2 is based on the assumption that the period-0 collateral constraint crosses the 45-degree line at a point located to the left of the initial level of debt,  $d_0$ . We now derive a condition under which such a crossing exists. The value of  $d$  at which the period-0 collateral constraint binds is a function of the initial level of debt,  $d_0$ . It is implicitly given by

$$\kappa \left[ y^T + \frac{(1-a)}{a} \left( y^T + \frac{d}{(1+r)} - d_0 \right)^{\frac{1}{\xi}} \right] = d.$$

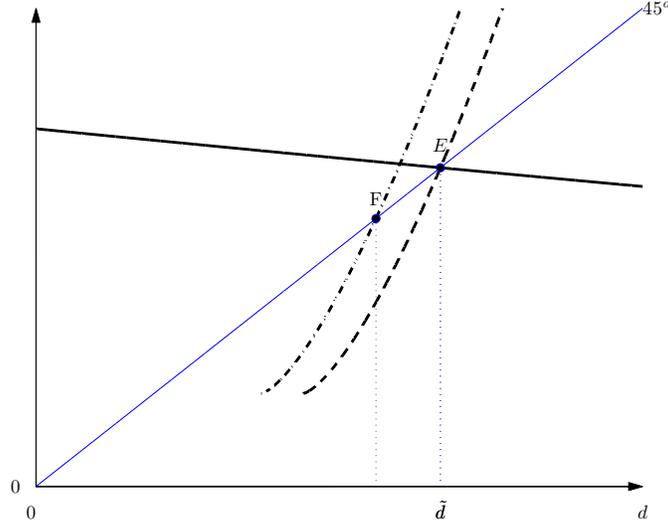
Clearly, this expression is satisfied at  $d = d_0 = \tilde{d}$ . Use the above equation to find the derivative of  $d$  with respect to  $d_0$  and evaluate it at  $d = d_0 = \tilde{d}$  to get

$$\frac{dd}{dd_0} = \frac{(1+r)S(\tilde{d}; \tilde{d})}{S(\tilde{d}; \tilde{d}) - 1}.$$

It follows that the period-0 collateral constraint will bind to the left of  $d_0$  when  $d_0$  takes values in a small neighborhood to the left of  $\tilde{d}$  if and only if the above derivative is greater than one, which, given that  $S(\tilde{d}; \tilde{d})$  is positive, happens if and only if  $S(\tilde{d}; \tilde{d}) > 1$ .

Figure 3 illustrates this result. It plots with a dashed line the right-hand side of the period-0 collateral constraint associated with  $d_0 = \tilde{d}$ . This line crosses the 45-degree line at

Figure 3: Existence of Multiple Equilibria



Notes. The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (21). The upward-sloping dashed and dashed-dotted lines are the right-hand sides of the period-0 collateral constraint, given in equation (23) for  $d_0 = \tilde{d}$  and  $d_0 < \tilde{d}$ , respectively. The figure is drawn under the assumptions that  $S(\tilde{d}; \tilde{d}) > 1$  and  $0 < \xi < 1$ .

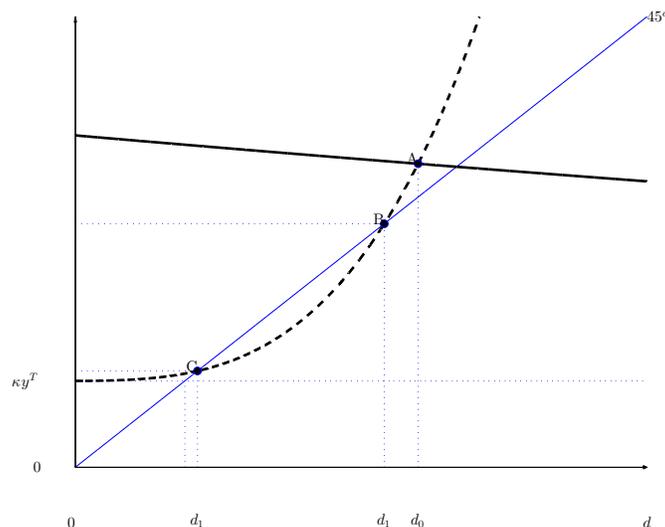
point  $E$ , where, by construction, it has a slope larger than 1. The figure also displays with a dash-dotted line the right-hand side of a period-0 collateral constraint associated with a value of  $d_0$  smaller than  $\tilde{d}$ . By continuity, if the decrease in  $d_0$  is sufficiently small, this line will cross the 45-degree line to the left of  $\tilde{d}$ , as shown by point  $F$  in the figure, guaranteeing the existence of a self-fulfilling financial-crises equilibrium.

In Appendix A we show that the condition  $S(\tilde{d}; \tilde{d}) > 1$  is indeed globally necessary and sufficient for the existence of a self-fulfilling financial-crisis equilibrium. Furthermore, there we characterize an interval containing all the initial values of debt associated with multiple equilibria. We summarize the main results of this section in the following proposition:

**Proposition 1 (Existence of Multiple Equilibria)** *Suppose  $y_t^N = 1$ ,  $y_t^T = y^T > 0$ ,  $r_t = r$ ,  $\beta(1+r) = 1$ , and  $\xi \in (0, 1)$ . Then, the steady-state equilibrium coexists with a self-fulfilling financial-crisis equilibrium if and only if  $S(\tilde{d}; \tilde{d}) > 1$  and  $d_0 \in [\hat{d}_0, \tilde{d}]$ , where  $S(\cdot; \cdot)$  is the slope of the right-hand side of the period-0 collateral constraint defined in equation (24),  $\hat{d}_0 \equiv \left(1 + \frac{\kappa}{1+r}\right) y^T - (1 - \xi) \left(\kappa \frac{1-a}{a} \frac{1}{1+r} \frac{1}{\xi}\right)^{\frac{1}{1-\xi}}$ , and  $\tilde{d}$  is defined in equation (22).*

**Proof:** See appendix A.

Figure 4: Two Self-Fulfilling Financial-Crisis Equilibria



Notes. The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (21). The upward-sloping broken line is the right-hand side of the period-0 collateral constraint, given in equation (23). The figure is drawn for the case that  $\xi < 1$ .

## 4.2 Multiple Self-Fulfilling Financial-Crisis Equilibria

The conditions given in Proposition 1 guarantee the existence of at least one self-fulfilling financial-crisis equilibrium. But there may be more. The right-hand side of the period-0 collateral constraint might cross the 45-degree line twice with a positive slope as shown in figure 4. The requirement of a positive slope ensures that at the second crossing consumption of tradables is positive in period 0 (see equation (24) and the comment immediately below it). In this case, each of the two crossings is a self-fulfilling financial-crisis equilibrium. These two equilibria coexist with the steady-state equilibrium. At points  $B$  and  $C$  in figure 4 the collateral constraint is binding in period 0 and  $d < d_0$ . The equilibrium associated with point  $C$  entails a larger drop in the value of collateral and more deleveraging in period 0 than the equilibrium associated with point  $B$ . This suggests that in the current environment self-fulfilling financial crises come in different sizes. Corollary 1 provides necessary and sufficient conditions for the existence of two self-fulfilling financial-crisis equilibria. It also provides the range of initial debt levels,  $d_0$ , for which multiple self-fulfilling financial-crisis equilibria exist.

**Corollary 1 (Existence of Two Self-Fulfilling Financial-Crisis Equilibria)** *Two self-fulfilling financial-crisis equilibria exist if and only if the conditions of Proposition 1 are satis-*

fixed. The range of initial debt levels,  $d_0$ , for which two self-fulfilling financial-crisis equilibria exist is  $\left[ \hat{d}_0, \min \left( \left( 1 + \frac{\kappa}{1+r} \right) y^T, \tilde{d} \right) \right)$ , where  $\hat{d}_0$  is defined in Proposition 1, and  $\tilde{d}$  is defined in equation (22).

**Proof:** See appendix B.

In words, Corollary 1 says that if there exist initial debt levels for which one self-fulfilling financial-crisis equilibrium exists, then there also exist initial debt levels for which two such equilibria exist.

### 4.3 High Intratemporal Elasticity of Substitution: $\xi > 1$

Multiplicity of equilibrium and the existence of self-fulfilling financial crises is not limited to the case of an intratemporal elasticity of substitution less than unity,  $\xi < 1$ . Figure 5 illustrates the existence of a self-fulfilling financial-crisis equilibrium with an intratemporal elasticity of substitution larger than unity. When  $\xi > 1$ , the right-hand side of the period-0 collateral constraint (shown with a broken line) is concave in  $d$ , and, as a result, there is at most one self-fulfilling financial crisis equilibrium (point  $B$  in the figure). The following proposition gives the necessary and sufficient condition for the existence of a self-fulfilling financial-crisis equilibrium when  $\xi > 1$ . It also provides the range of initial levels for debt,  $d_0$ , for which such an equilibrium exists.

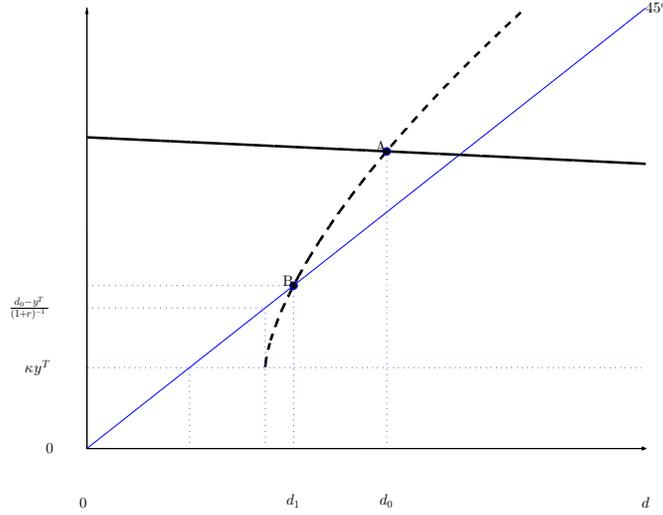
**Proposition 2 (Existence of Multiple Equilibria When  $\xi > 1$ )** *Suppose  $y_t^N = 1$ ,  $y_t^T = y^T > 0$ ,  $r_t = r$ ,  $\beta(1+r) = 1$ , and  $\xi > 1$ . Then, the steady-state equilibrium coexists with a self-fulfilling financial-crisis equilibrium if and only if  $S(\tilde{d}; \tilde{d}) > 1/\xi$  and  $d_0 \in \left( \left( 1 + \frac{\kappa}{1+r} \right) y^T, \tilde{d} \right)$ , where  $S(\cdot; \cdot)$  is the slope of the right-hand side of the period-0 collateral constraint defined in equation (24) and  $\tilde{d}$  is defined in equation (22).*

**Proof:** See appendix C.

### 4.4 Discussion

The intuition behind the existence of self-fulfilling financial-crisis equilibria is as follows. Imagine the economy being originally in a steady state with debt constant and equal to  $d_0$ . Unexpectedly, the public becomes pessimistic and aggregate demand contracts. The contraction in aggregate demand means that households want to consume less of both types of good, tradable and nontradable. Tradables can always be sold abroad, but nontradables must be sold exclusively in the domestic market. Thus, the fall in the demand for consumption goods causes a decline in the relative price of nontradables,  $p_0$ . As a result, the value of

Figure 5: Self-Fulfilling Financial-Crisis Equilibrium When  $\xi > 1$

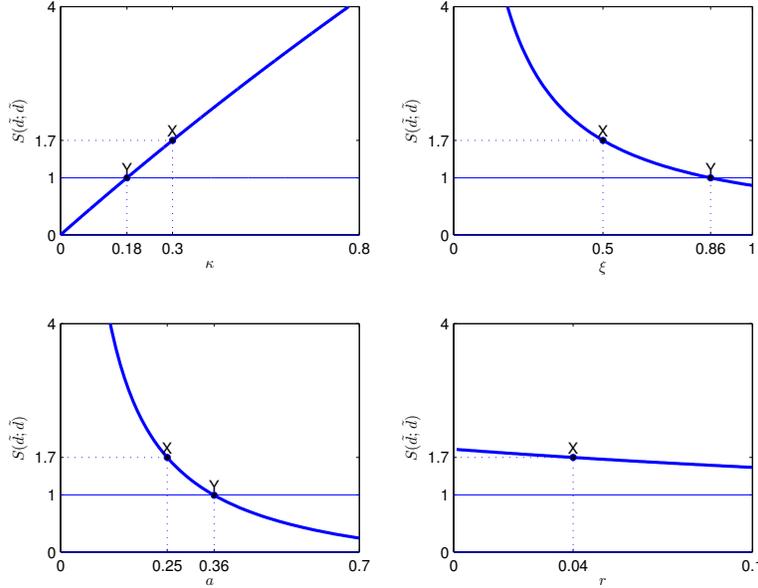


Notes. The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (21). The upward-sloping broken line is the right-hand side of the period-0 collateral constraint, given in equation (23).

collateral, given by  $\kappa(y^T + p_0 y^N)$ , also falls. The reduction in collateral is so large that it forces households to deleverage. The generalized decline in the value of collateral represents the quintessential element of a financial crisis. To reduce their net debt positions, households must cut spending, validating the initial pessimistic sentiments, and making the financial crisis self-fulfilling. The contraction in the debt position and the fall in the relative price of nontradables imply that the self-fulfilling financial crisis occurs in the context of a current account surplus and a depreciation of the real exchange rate.

Although the model studied in this section is highly stylized, it is of interest to see whether self-fulfilling financial crises exist for reasonable parameterizations. Quantitative models of open economies with collateral constraints calibrated to emerging countries assume debt limits of about 30 percent of an annual GDP, which implies a value of  $\kappa$  of 0.3. Estimates of the intratemporal elasticity of substitution between tradables and nontradables in emerging countries typically lie around 1/2 (Akinci, 2011). The parameter  $a$ , the weight on tradable consumption in the CES aggregator, is typically set at around 1/4, which implies that if the aggregator were of the Cobb-Douglas form ( $\xi = 1$ ), the share of tradables in total consumption would be 25 percent. The world interest rate is frequently calibrated to 4 percent per year, or  $r = 0.04$ . Finally, we assume that the endowment of tradables is equal to 1. With these values in hand, one can calculate the slope  $S(\tilde{d}; \tilde{d})$  by using (22) to find  $\tilde{d}$  and then using this value to evaluate (24) at  $d_0 = d = \tilde{d}$ . This yields  $S(\tilde{d}; \tilde{d}) = 1.7$ . A slope

Figure 6: Existence of Multiple Equilibria for Different Parameterizations of the Model



Notes.  $X$  baseline parameterization;  $Y$  value at which  $S(\tilde{d}; \tilde{d})$  takes the value 1. The model displays multiple equilibria if  $S(\tilde{d}; \tilde{d}) > 1$ . In each panel, all parameters other than the one displayed on the horizontal axis are fixed at their baseline values ( $\kappa = 0.3$ ,  $\xi = 0.5$ ,  $a = 0.25$ ,  $r = 0.04$ ).

larger than unity implies the existence of self-fulfilling financial-crisis equilibria. This result suggests that self-fulfilling crises can arise for empirically plausible parameterizations of the model.

Figure 6 explores the existence of self-fulfilling financial-crisis equilibria around the present calibration. Each panel displays the value of  $S(\tilde{d}; \tilde{d})$  as a function of a particular parameter, holding all other parameters at their baseline values. The top-left panel shows that the slope  $S(\tilde{d}; \tilde{d})$  is increasing in  $\kappa$  and crosses the threshold of unity at  $\kappa = 0.18$ . This suggests that the emergence of multiple equilibria is more likely the higher  $\kappa$  is. This result is intuitive, because  $\kappa$  represents the fraction of income that is pledgeable as collateral. Thus,  $\kappa$  captures the sensitivity of collateral with respect to income. The top-right panel of the figure shows that the less substitutable tradables and nontradables are, the more likely it is that self-fulfilling financial-crisis equilibria exist. Intuitively, the smaller is the intratemporal elasticity of substitution  $\xi$ , the larger will be the increase in the relative price of nontradables,  $p$ , required to clear the market in response to an increase in desired absorption. In turn, because  $p$  determines the value of collateral, we have that the smaller  $\xi$  is, the steeper is the slope of the collateral constraint. Multiple equilibria exist for values of  $\xi$  larger lower than 0.86. The lower-left panel shows that multiple equilibria become more likely the smaller the share parameter  $a$  is, with a threshold of 0.36. The reason is that the ratio  $(1 - a)/a$  acts like

a shifter of the demand for nontradables,  $p = (1 - a)/a(c^N/c^T)^{-1/\xi}$ . The larger the home bias  $1 - a$  is, the larger the shifter will be. This means that as the home bias increases so does the sensitivity of  $p$  with respect to desired absorption. Finally, as shown in the bottom-right panel of the figure, for the present model specification multiplicity of equilibrium appears to be relatively insensitive to changes in the world interest rate,  $r$ .

## 5 Sunspots and Persistent Financial Crises

In the perfect-foresight economy studied in section 4, self-fulfilling financial crises last for only one period. Multi-period crises equilibria do not exist. To see this, suppose, on the contrary, that the economy suffers self-fulfilling financial crises in periods  $t$  and  $t + 1$ . Use equation (16) to replace  $y_t^T + \frac{d_{t+1}}{1+r} - d_t$  by  $c_t^T$  in (17) holding with equality and solve for  $c_t^T$  as an increasing function of  $d_{t+1}$ :

$$c_t^T = \left[ \left( \frac{d_{t+1}}{\kappa} - y^T \right) \frac{a}{1 - a} \right]^\xi. \quad (25)$$

A similar expression holds in  $t + 1$ , since the collateral constraint binds in that period as well,

$$c_{t+1}^T = \left[ \left( \frac{d_{t+2}}{\kappa} - y^T \right) \frac{a}{1 - a} \right]^\xi.$$

From the analysis of previous sections, we know that if the economy is in a financial crisis in  $t + 1$ , it deleverages, that is  $d_{t+2} < d_{t+1}$ . This implies, from the above two equations, that  $c_{t+1}^T < c_t^T$ . But this is impossible in equilibrium, because according to the Euler equation (15), it would require  $\mu_t < 0$ , violating the nonnegativity requirement of this Lagrange multiplier, equation (19).

The predicted one-period life of financial crises is at odds with observed episodes of financial duress, which are typically multi-period phenomena. In this section, we show that in a setting with nonfundamental uncertainty self-fulfilling financial crises can be persistent. To establish this result, we characterize a two-period self-fulfilling financial crisis. The analysis, however, can be extended to longer lasting crises. Assume that  $\xi \in (0, 1)$ . The economy is the same infinite-horizon environment studied in section 4, with one modification. Suppose there is an exogenous random variable  $s_t$  that takes on the values 1 or 0. If  $s_t$  takes the value 1, then consumers feel pessimistic, and if  $s_t$  takes on the value 0, then agents have an optimistic outlook. The variable  $s_t$  is known as a sunspot because its sole role is to coordinate agents' expectations.

The economy starts with pessimistic sentiments, so that  $s_0 = 0$ . In period 1,  $s_t$  takes

the value 1 with probability  $\pi$  and the value 0 with probability  $1 - \pi$ , where  $\pi \in (0, 1)$  is a parameter. Suppose that pessimism lasts for at most 2 periods, so that  $s_t = 0$  for all  $t \geq 2$ . We wish to show that there exists a probability distribution of  $s_1$ , that is, a value of  $\pi$ , that can support a two-period self-fulfilling financial crisis as a rational expectations equilibrium. We define a two-period self-fulfilling financial crisis equilibrium as an equilibrium in which the collateral constraint binds in periods 0 and 1. We focus on equilibria in which the economy reaches a steady state in period 2. We establish this result by construction.

The level of debt in period 1 is determined by the collateral constraint (17) holding with equality, that is,

$$d_1 = \kappa \left[ y^T + \frac{(1-a)}{a} \left( y^T + \frac{d_1}{(1+r)} - d_0 \right)^{\frac{1}{\xi}} \right].$$

From section 4, we know that this equation yields a positive real value of  $d_1$  under the assumptions that  $S(\tilde{d}; \tilde{d}) > 1$  and  $d_0 \in (\hat{d}_0, \tilde{d})$ , which we maintain. Furthermore, the analysis presented in section 4 shows that the economy deleverages in period 0, that is,

$$d_1 < d_0.$$

Consumption is guaranteed to be positive (by the assumption  $d_0 \in (\hat{d}_0, \tilde{d})$ ) and given by the resource constraint (16)

$$c_0^T = y^T + \frac{d_1}{1+r} - d_0.$$

In period 1, the equilibrium levels of debt and consumption depend on the realization of the sunspot variable  $s_1$ . Let  $d_{t+1,i}$  and  $c_{t,i}^T$  denote the levels of debt and consumption for  $t \geq 1$  if  $s_1 = i$  for  $i = 0, 1$ .

If  $s_1 = 0$ , then the economy reaches a steady state with  $d_{t+1,0} = d_1$  and  $c_{t,0}^T = c_{1,0}^T$ , for all  $t \geq 1$ , where

$$c_{1,0}^T = y^T - \frac{r}{1+r} d_1.$$

The above three expressions imply that

$$c_{1,0}^T > c_0^T.$$

If  $s_1 = 1$ , the economy experiences a self-fulfilling financial-crisis equilibrium in period 1, with a binding collateral constraint in period 1 and a steady state starting in period 2. From the analysis presented in section 4 we know that such an equilibrium exists if  $d_1 > \hat{d}_0$ . This will be the case if  $d_0$  is sufficiently close to  $\tilde{d}$ . Furthermore, since when  $s_1 = 1$ , the collateral constraint binds in periods 0 and 1, we have, from the analysis at the beginning

of this section, that consumption must decline between periods 0 and 1, that is,

$$c_{1,1}^T < c_0^T.$$

This construction guarantees that all equilibrium conditions (equations (15)-(20)) are satisfied for all  $t \geq 0$  and  $s_1 = 0, 1$ , except for the Euler equation (15) in period 0. Thus, as the final step of this proof, we show that one can pick  $\pi$  to ensure that the Euler equation holds in period 0. This equation is given by

$$[1 - (1 + r)\mu_0]\Lambda(c_0^T) = \pi\Lambda(c_{1,1}^T) + (1 - \pi)\Lambda(c_{1,0}^T).$$

Since we have already determined the entire path of consumption, this is one equation in one unknown,  $\mu_0$ . Thus, the existence of this equilibrium hangs on the existence of values of  $\pi \in (0, 1)$  that guarantee a nonnegative value of  $\mu_0$ . This is indeed the case, because  $c_{1,1}^T < c_0^T < c_{1,0}^T$  and because  $\Lambda(\cdot)$  is a decreasing function. In fact, there is a range of values of  $\pi$  that make  $\mu_0 \geq 0$ , which is given by

$$\pi \in (0, \pi^*],$$

where

$$\pi^* \equiv \frac{\Lambda(c_0^T) - \Lambda(c_{1,0}^T)}{\Lambda(c_{1,1}^T) - \Lambda(c_{1,0}^T)} \in (0, 1).$$

According to this expression, in order for the possibility that the financial crisis extends for two periods it is necessary that households assign a sufficiently high probability (greater than  $1 - \pi^*$ ) to the event that the economy will emerge from the crisis in the second period ( $t = 1$ ). Moreover, the higher the chances households place on getting out of the crisis in period 1, the more easily the conditions for a two-period crisis to exist are satisfied. This might seem paradoxical. However, because expectations are rational, it is also the case that the higher the probability agents assign to exiting the crisis in period 1, the less likely it will be that the crisis will last for more than one period.

## 6 Underborrowing

The pecuniary externality created by the presence of the relative price of nontradables in the collateral constraint induces an allocation that is in general suboptimal, compared to the best allocation possible among all of the ones that satisfy the collateral constraint, the resource constraint for tradable goods, and the equilibrium conditions of the market for nontradable

goods. The standard result stressed in the related literature is that the unregulated economy overborrows. That is, external debt is higher than it would be if households internalized the pecuniary externality. We say that an economy underborrows (overborrows) if its net external debt is on average lower (higher) in the unregulated competitive equilibrium than in the constrained social planner's allocation. The unregulated competitive equilibrium is the solution to equations (10)-(14). The constrained social planner's allocation is the pair of processes  $\{c_t^T, d_{t+1}\}_{t=0}^{\infty}$  that maximizes  $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N))$  subject to (11), (12), and a no-Ponzi-game constraint. Because of the wedge it introduces between the allocation associated with the unregulated competitive equilibrium and the social planner's allocation, the collateral constraint opens the door to welfare improving policy intervention.

This section accomplishes two tasks. First, it addresses the question of whether overborrowing continues to obtain in collateral-constrained economies exhibiting multiple equilibria. Second, it characterizes a fiscal instrument that supports the social planner's allocation as a competitive equilibrium. We begin with the second of these tasks. The fiscal instrument we consider is capital controls. This type of fiscal policy is of interest for two reasons. First, as we will see, the optimal capital control policy fully internalizes the pecuniary externality, in the sense that it induces the representative household to behave as if it understood that its own borrowing choices influence the relative price of nontradables and therefore the value of collateral. Second, capital controls are of interest because they represent a tax on external borrowing, which is the variable most directly affected by the pecuniary externality.

Let  $\tau_t$  be a proportional tax on debt acquired in period  $t$ . If  $\tau_t$  is positive, it represents a proper capital control tax, whereas if it is negative it has the interpretation of a borrowing subsidy. The revenue from capital control taxes is given by  $\tau_t d_{t+1}/(1+r_t)$ . We assume that the government consumes no goods and that it rebates all revenues from capital controls to the public in the form of lump-sum transfers (lump-sum taxes if  $\tau_t < 0$ ), denoted  $\ell_t$ .<sup>1</sup> The budget constraint of the government is then given by

$$\tau_t \frac{d_{t+1}}{1+r_t} = \ell_t. \quad (26)$$

The household's sequential budget constraint now becomes

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + (1 - \tau_t) \frac{d_{t+1}}{1+r_t} + \ell_t.$$

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<sup>1</sup>The results would be unchanged if one were to assume alternatively that revenues from capital control taxes are rebated by means of a proportional income transfer. Since tradable and nontradable income is exogenous to the household, this transfer would be nondistorting and therefore equivalent to a lump-sum transfer.

It is apparent from this expression that the capital control tax distorts the borrowing decision of the household. In particular, the gross interest rate on foreign borrowing perceived by the private household is no longer  $1 + r_t$ , but  $(1 + r_t)/(1 - \tau_t)$ . All other things equal, the higher is  $\tau_t$ , the higher is the interest rate perceived by households. Thus, by changing  $\tau_t$  the government can encourage or discourage borrowing. All optimality conditions associated with the household's optimization problem (equations (5)-(9)) are unchanged, except for the debt Euler equation (7), which now takes the form

$$\left(\frac{1 - \tau_t}{1 + r_t} - \mu_t\right) \lambda_t = \beta \mathbb{E}_t \lambda_{t+1}.$$

A competitive equilibrium in the economy with capital control taxes is then a set of processes  $c_t^T$ ,  $d_{t+1}$ ,  $\lambda_t$ ,  $\mu_t$ , and  $p_t$  satisfying

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1 + r_t}, \quad (27)$$

$$d_{t+1} \leq \kappa [y_t^T + p_t y^N], \quad (28)$$

$$\lambda_t = U'(A(c_t^T, y^N)) A_1(c_t^T, y^N), \quad (29)$$

$$\left(\frac{1 - \tau_t}{1 + r_t} - \mu_t\right) \lambda_t = \beta \mathbb{E}_t \lambda_{t+1}, \quad (30)$$

$$p_t = \frac{A_2(c_t^T, y^N)}{A_1(c_t^T, y^N)}, \quad (31)$$

$$\mu_t [\kappa (y_t^T + p_t y^N) - d_{t+1}] = 0, \quad (32)$$

$$\mu_t \geq 0, \quad (33)$$

given a policy process  $\tau_t$ , exogenous driving forces  $y_t^T$  and  $r_t$ , and the initial condition  $d_0$ .

The benevolent government sets capital control taxes to maximize the household's lifetime utility subject to the restriction that the optimal allocation be supportable as a competitive equilibrium. It follows that all of the above competitive equilibrium conditions are constraints of the government's optimization problem. Formally, the optimal competitive equilibrium is a set of processes  $\tau_t$ ,  $c_t^T$ ,  $d_{t+1}$ ,  $\lambda_t$ ,  $\mu_t$ , and  $p_t$  that solve the problem of maximizing

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N)) \quad (34)$$

subject to (27)-(33), given processes  $y_t^T$  and  $r_t$  and the initial condition  $d_0$ . In the welfare function (34), we have replaced consumption of nontradables,  $c_t^N$ , with the endowment of

nontradables,  $y^N$ , because the planner takes into account that in a competitive equilibrium the market for nontradables clears at all times. Equilibrium conditions (27)-(33) can be reduced to two expressions. Specifically, processes  $c_t^T$  and  $d_{t+1}$  satisfy equilibrium conditions (27)-(33) if and only if they satisfy (27) and

$$d_{t+1} \leq \kappa \left[ y_t^T + \frac{1-a}{a} \left( \frac{c_t^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right]. \quad (35)$$

We establish this result in Appendix D.

We can then state the government's problem as

$$\max_{\{c_t^T, d_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N)) \quad (34)$$

subject to

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1+r_t}, \quad (27)$$

$$d_{t+1} \leq \kappa \left[ y_t^T + \frac{1-a}{a} \left( \frac{c_t^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right]. \quad (35)$$

Comparing the levels of debt in the optimal competitive equilibrium and in the unregulated equilibrium (i.e., the equilibrium without government intervention), we can determine whether the lack of optimal government intervention results in overborrowing or underborrowing.

Consider the optimal allocation in the perfect-foresight economy analyzed in section 4. Suppose that the initial value of debt,  $d_0$ , satisfies  $d_0 \in (\hat{d}_0, \tilde{d})$ , so that self-fulfilling financial-crisis equilibria exist, as shown in figure 2. Since one possible competitive equilibrium is  $d_t = d_0$  (point A in figure 2) with  $c_t^T = y^T - rd_0/(1+r)$  for all  $t \geq 0$ , and since this equilibrium is the first best equilibrium (i.e., the equilibrium that would result in the absence of the collateral constraint), it also has to be the optimal competitive equilibrium. The capital control tax associated with the optimal equilibrium can be deduced from inspection of equation (44). Because consumption of tradables is constant over time and because in this analytical example  $\beta(1+r) = 1$ , it follows that  $\tau_t = 0$  for all  $t \geq 0$ .

Compare now the level of debt in the optimal allocation with the level of debt associated with the unregulated competitive equilibrium. Does the economy overborrow or underborrow? The answer to this question depends on which of the multiple equilibria materializes (point A or point B in figure 2). Suppose the unregulated competitive equilibrium happens to be the one in which the collateral constraint binds in period 0, point B in figure 2. In this

case the unregulated economy underborrows at all times, since the level of debt at point B is less than the optimal level of debt,  $d_0$ . If, on the other hand, the unregulated competitive equilibrium happens to be the unconstrained equilibrium (point A in the figure), then there is neither underborrowing nor overborrowing, since its associated level of debt coincides with the optimal level,  $d_0$ . Thus, in this economy, there is either underborrowing or optimal borrowing, depending on whether the competitive equilibrium happens to be the constrained or the unconstrained one.

Similarly, in the economy with two self-fulfilling financial-crisis equilibria depicted in figure 4 the optimal equilibrium is at point A, with constant consumption and capital control taxes equal to zero at all times. If the unregulated economy coordinates on equilibrium B or equilibrium C, then there is underborrowing, and if it coordinates on equilibrium A, then there is neither underborrowing nor overborrowing. Finally, the underborrowing result is robust to the introduction of extrinsic uncertainty. In the sunspot economy of section 5, the economy underborrows in every period in which the sunspot variable makes agents coordinate on a self-fulfilling financial-crisis equilibrium.

## 7 Implementation

The optimal policy is mute with regard to equilibrium implementation. In the context of the economy studied in section 6, this means that the optimal policy  $\tau_t = 0$  does not guarantee that the competitive equilibrium will be the optimal one (e.g., point A in figures 2 and 4). In particular, the policy rule  $\tau_t = 0$  for all  $t$  may result in an unintended competitive equilibrium, like point B in figure 2 or points B or C in figure 4. Thus a policy of setting  $\tau_t = 0$  at all times may fail to implement the optimal allocation. However, any capital-control policy that succeeds in implementing the optimal allocation must deliver  $\tau_t = 0$  for all  $t$  in equilibrium. The difference between a policy that sets  $\tau_t = 0$  under all circumstances and a policy that implements the optimal allocation does not lie in the capital control tax that results in equilibrium, but in the tax rates that would be imposed off equilibrium.

To shed light on the issue of implementation, here we study a capital-control feedback rule that implements the optimal equilibrium in the model economy of section 4. Specifically, consider the capital control policy

$$\tau_t = \tau(d_{t+1}, d_t) \tag{36}$$

satisfying  $\tau(d, d) = 0$ . To see whether this capital-control policy is consistent with the optimal equilibrium, it suffices to verify that the Euler equation is satisfied since this is the

only equilibrium condition in which  $\tau_t$  appears. Under the tax-policy rule (36), the Euler equation in period 0 is given by

$$\frac{\Lambda(c_0^T)}{\Lambda(c_1^T)} = \frac{1}{1 - \tau(d_1, d_0) - (1 + r)\mu_0}. \quad (37)$$

In the optimal equilibrium, we have that  $c_1^T/c_0^T = 1$ , that  $d_1 = d_0$  (which implies that  $\tau(d_1, d_0) = 0$ ), and that  $\mu_0 = 0$ , so the Euler equation holds. This establishes that the proposed policy is consistent with the optimal allocation.

In addition to supporting the optimal equilibrium, if appropriately designed, the tax policy (36) can rule out the unintended equilibria. Recalling that  $c_0^T$  and  $c_1^T$  satisfy  $c_0^T = y^T + d_1/(1 + r) - d_0$  and  $c_1^T = y^T - rd_1/(1 + r)$ , we can write the Euler equation (37) as

$$\frac{\Lambda(y^T + d_1/(1 + r) - d_0)}{\Lambda(y^T - r/(1 + r)d_1)} = \frac{1}{1 - \tau(d_1, d_0) - (1 + r)\mu_0}. \quad (38)$$

Now pick the function  $\tau(\cdot, \cdot)$  in such a way that if a self-fulfilling financial-crisis equilibrium occurs and the economy deleverages, then the Euler equation holds only if  $\mu_0$  is negative. Specifically, set  $\tau(d_1, d_0)$  to satisfy

$$\frac{\Lambda(y^T + d_1/(1 + r) - d_0)}{\Lambda(y^T - r/(1 + r)d_1)} < \frac{1}{1 - \tau(d_1, d_0)},$$

for all  $d_1 < d_0$ . Clearly, this policy requires  $\tau(d_1, d_0) > 0$  if  $d_1 < d_0$ . Under this capital control policy, the Euler equation would not hold for any value of  $d_1$  less than  $d_0$ , since it would require  $\mu_0 < 0$ , which violates the nonnegativity constraint (33). This means that any equilibrium in which the economy deleverages is ruled out.

The capital control policy that rules out self-fulfilling financial-crisis equilibria and ensures that only the optimal equilibrium emerges is one in which the policy maker is committed to imposing capital control taxes in the case of capital outflows, that is, in the event that  $d_1 < d_0$ . This type of capital control policy serves as a metaphor for a variety of policies that are often contemplated in emerging countries during financial panics and that aim at temporarily restricting capital outflows, including restrictions on foreign exchange markets and profit and dividend repatriations. In the present perfect-foresight economy, the mere threat of the imposition of capital control taxes in the event of capital flights suffices to fend off self-fulfilling financial crises. In equilibrium, these threats never need to be carried out.

## 8 An Economy with Fundamental Uncertainty

We now characterize numerically the debt dynamics in a stochastic version of the economy presented in section 2. To this end, we assume a joint stochastic process for the tradable endowment and the country interest rate and calibrate the structural parameters of the model to match certain features of the Argentine economy. The focus of this section is to illustrate that under different plausible parameterizations the economy may exhibit equilibria with underborrowing or with overborrowing.

### 8.1 Calibration

We calibrate the model at a quarterly frequency. Table 1 summarizes the calibration. We set  $\kappa$  so that the upper limit of net external debt is 30 percent of annual output. This value is in line with those used in the quantitative literature on output-based collateral constraints (e.g., Bianchi, 2011). Because the time unit in the model is a quarter, this calibration restriction implies a value of  $\kappa$  of 1.2 ( $= 0.3 \times 4$ ). The calibration of the remaining parameters follows Schmitt-Grohé and Uribe (2016). We set  $\beta = 0.9635$ ,  $\sigma = 1/\xi = 2$ ,  $a = 0.26$ , and  $y^N = 1$ . The exogenous variables  $y_t^T$  and  $r_t$  are assumed to follow a bivariate AR(1) process of the form

$$\begin{bmatrix} \ln y_t^T \\ \ln \frac{1+r_t}{1+r} \end{bmatrix} = A \begin{bmatrix} \ln y_{t-1}^T \\ \ln \frac{1+r_{t-1}}{1+r} \end{bmatrix} + \epsilon_t, \quad (39)$$

where  $\epsilon_t \sim N(\emptyset, \Sigma_\epsilon)$ . Schmitt-Grohé and Uribe (2016) estimate this process on Argentine quarterly data over the period 1983:Q1 to 2001:Q4. The estimated parameters are

$$A = \begin{bmatrix} 0.79 & -1.36 \\ -0.01 & 0.86 \end{bmatrix}; \quad \Sigma_\epsilon = \begin{bmatrix} 0.00123 & -0.00008 \\ -0.00008 & 0.00004 \end{bmatrix}; \quad r = 0.0316.$$

### 8.2 Equilibrium Approximation

To approximate the equilibrium, we develop an Euler equation iteration procedure over a discretized state space. The economy possesses two exogenous states,  $y_t^T$  and  $r_t$ , and one endogenous state,  $d_t$ . We discretize  $\ln y_t^T$  using 21 evenly spaced points centered at 0 and  $\ln(1+r_t)/(1+r)$  using 11 evenly spaced points centered at 0. The upper bound of the grids of  $\ln y_t^T$  and  $\ln((1+r_t)/(1+r))$  are taken to be  $\sqrt{10}$  times the corresponding unconditional standard deviations implied by the estimated version of the VAR system (39). The resulting intervals are  $[\ln \underline{y}^T, \ln \bar{y}^T] = [-0.3858, 0.3858]$  and  $[\ln(\frac{1+r}{1+r}), \ln(\frac{1+r}{1+r})] = [-0.0539, 0.0539]$ .

Table 1: Calibration of the Stochastic Economy

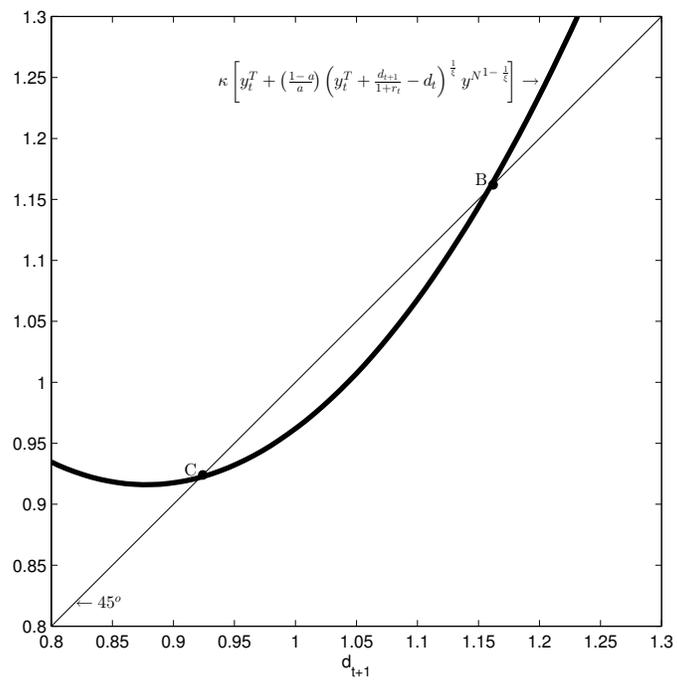
Parameter	Value	Description
$\kappa$	1.2	Parameter of collateral constraint
$\sigma$	2	Inverse of intertemporal elasticity of consumption
$\beta$	0.9635	Quarterly subjective discount factor
$r$	0.0316	Steady state quarterly country interest rate
$\xi$	0.5	Elasticity of substitution between tradables and nontradables
$a$	0.26	Parameter of CES aggregator
$y^N$	1	Nontradable output
$y^T$	1	Steady-state tradable output
Discretization of State Space		
$n_{y^T}$	21	Number of grid points for $\ln y_t^T$ , equally spaced
$n_r$	11	Number of grid points for $\ln[(1+r_t)/(1+r)]$ , equally spaced
$n_d$	501	Number of grid points for $d_t$ , equally spaced
$[\ln y^T, \ln \bar{y}^T]$	[-0.3858, 0.3858]	Range for tradable output
$[\ln(\frac{1+r}{1+r}), \ln(\frac{1+\bar{r}}{1+r})]$	[-0.0539, 0.0589]	Range for interest rate
$[\underline{d}, \bar{d}]$	[0, 3.5]	Debt range

Note. The time unit is one quarter.

We compute the transition probability matrix using the simulation approach of Schmitt-Grohé and Uribe (2009). For the endogenous state variable,  $d_t$ , we use 501 equally spaced points in the interval  $[\underline{d}, \bar{d}] = [0, 3.5]$ .

As in the analysis of section 4, the present economy features an equilibrium collateral constraint whose right-hand side may intersect the 45-degree line twice with a positive slope, implying that the set of values  $d_{t+1}$  that satisfy both the period- $t$  resource constraint and the period- $t$  collateral constraint may not be convex. For example, figure 7 displays the value of collateral as a function of  $d_{t+1}$  for the state  $(y_t^T, r_t, d_t) = (0.7633, 0.0541, 1.5960)$ . In this state, there are two disjoint sets of  $d_{t+1}$  for which the collateral constraint is satisfied. In between these two sets, the price of nontradables is too low to guarantee the satisfaction of the borrowing limit. As we saw in the analytical part of the paper, this type of constraint can give rise to financial crises. Agents protect themselves against these eventualities by exercising precautionary savings. In turn, precautionary savings places the economy in a region of the state space in which the right-hand side of the collateral constraint crosses the 45-degree line less often. As it turns out atomistic agents engage in too much precautionary savings relative to a planner who internalizes the effect that borrowing has on the value of collateral. In the stochastic environment we study here, the economy fluctuates virtually all of the time in a region of the state space in which the collateral constraint has a slope

Figure 7: Multiple Binding Debt Levels In the Stochastic Economy



Note. The value of collateral is evaluated at the state  $(y_t^T, r_t, d_t) = (0.7633, 0.0541, 1.5960)$ . All parameters take the values indicated in table 1.

bigger than unity, that is, in a region in which when the economy borrows one more unit the borrowing capacity increases by more than one unit. This is the case both in the unregulated and social planner equilibria. In terms of figure 4, the economy is virtually all the time fluctuating in the region to the right of point  $B$ . There, the collateral constraint is slack, but its slope is larger than one. The difference between the unregulated and regulated economies is that the social planner, by internalizing the pecuniary externality, is able to steer clear of a binding collateral constraint more often than the unregulated economy does.

The numerical solution must take a stance on how to handle the possibility of indeterminacy of the rational expectations equilibrium of the type identified in section 4. Failing to address this issue may result in nonconvergence of numerical algorithms. We focus on two canonical equilibrium selection mechanisms suggested by the preceding theoretical analysis. We label these mechanisms (b) and (c) to indicate their relation to the corresponding points in figure 4. The selection mechanisms are defined as follows:

Selection mechanism (b): If for a given current state  $(y_t^T, r_t, d_t)$  there are one or more values of  $d_{t+1}$  for which all equilibrium conditions are satisfied pick the largest one for which the collateral constraint is binding.

Selection mechanism (c): If for a given current state  $(y_t^T, r_t, d_t)$  there are one or more values of  $d_{t+1}$  for which all equilibrium conditions are satisfied pick the smallest one for which the collateral constraint is binding.

### 8.3 Underborrowing

Figure 8 displays the unconditional distribution of external debt,  $d_t$ .<sup>2</sup> The different equilibrium selection criteria give rise to different debt distributions, revealing the presence of multiple equilibria. The more pessimistic equilibrium selection criterion (c) (dash-dotted line in the figure), which favors larger self-fulfilling debt crises, yields a debt distribution with a mean of 12.0 percent of annual output. The distribution of debt associated with selection criterion (b) (the dotted line in figure 8) is located to the right of the one associated with criterion (c). Although the difference is not large—the average annual debt-to-output ratio is 0.3 percentage points higher under criterion (b)—it can lead to computational difficulties if ignored. Specifically, if one were to attempt to compute the equilibrium assuming uniqueness, standard Euler-equation iteration procedures will in general not converge.

In section 6 we showed that when the collateral constraint opens the door to multiple

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<sup>2</sup>To avoid clutter, the densities are smoothed out as follows. For each grid point  $d_i$  the associated smoothed density is the average of the densities associated with points  $d_{i-20}$  to  $d_i$  for  $i = 21, \dots, 501$ . Thus the smoothed density has 481 bins.

equilibria, the unregulated competitive equilibrium can display underborrowing, in the sense that the level of external debt is below the optimal allocation achieved by a benevolent social planner with access to a capital-control tax. Figure 8 shows that this result also obtains in a stochastic economy under a plausible calibration.

The optimal allocation is relatively easy to compute because the planner’s problem can be cast in the form of a Bellman equation problem. Specifically, the recursive version of the planner’s problem of maximizing (34) subject to (27) and (35) is given by

$$v(y^T, r, d) = \max_{c^T, d'} \left\{ U(A(c^T, y^N)) + \beta \mathbb{E} \left[ v(y^{T'}, r', d') \mid y^T, r \right] \right\}$$

subject to

$$c^T + d = y^T + \frac{d'}{1+r}$$

$$d' \leq \kappa \left[ y^T + \frac{1-a}{a} \left( \frac{c^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right],$$

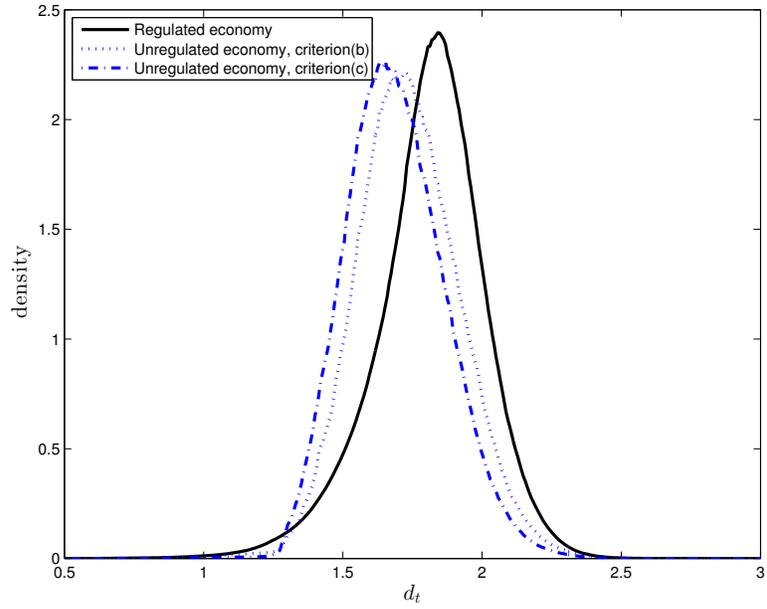
where a prime superscript denotes next-period values. Although the constraints of this control problem may not represent a convex set in tradable consumption and debt, the fact that the optimal allocation is the result of a utility maximization problem, implies that its solution is generically unique. The calibration of the economy is the same as that used for the unregulated economy, summarized in table 1.

Figure 8 displays with a solid line the unconditional distribution of net external debt,  $d_t$ , under optimal policy. The unregulated economy displays underborrowing, in the sense that its debt distribution lies to the left of the one associated with the optimal capital control policy. The average annual debt-to-output ratio in the regulated economy is 13.1 percentage points compared with 12.3 and 12.0 percentage points in the unregulated economies (b) and (c), respectively. In the unregulated economy, households have an incentive to over self insure. This is due to the fact that the unregulated economy is fragile as it is more prone to financial crises caused by a binding collateral constraint.

## 8.4 Overborrowing

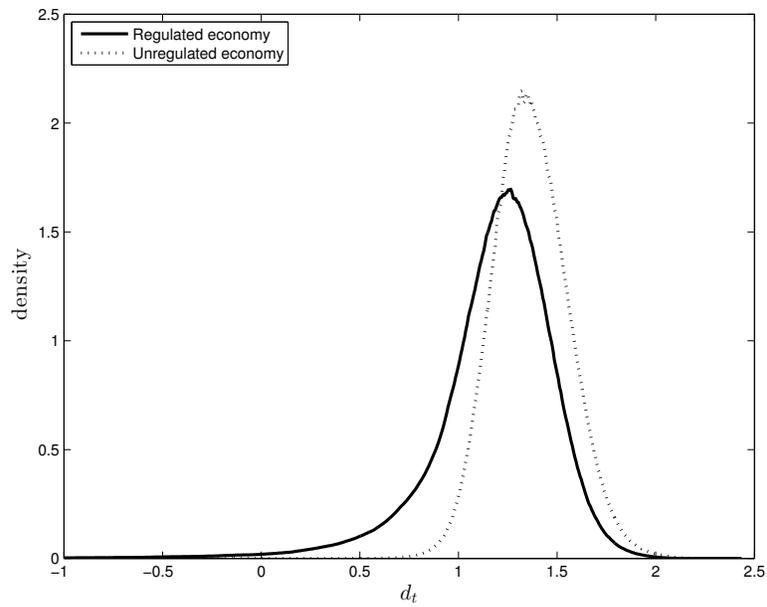
The analysis of subsection 4.4 suggests that multiplicity of equilibrium is less likely the lower is the leverage limit  $\kappa$ , the higher is the elasticity of intratemporal substitution between tradable and nontradable consumption,  $\xi$ , and the higher is the share parameter  $a$  in the CES aggregator. To illustrate this point, we present a plausible calibration for which the standard overborrowing result obtains. Specifically, we lower  $\kappa$  from 1.2 to 0.8, and increase

Figure 8: Equilibrium Underborrowing



Note. Replication program plotd.m.

Figure 9: Equilibrium Overborrowing



$\xi$  and  $a$  from 0.5 and 0.26 to 0.85 and 0.5, respectively. All other structural parameters take the values shown in table 1. The debt grid ranges from -4 to 2.5 and contains 1,001 equally spaced points. We increase the number of debt grid points from 501 to 1,001 to maintain a step size that is comparable to the one used in the baseline case as the debt range is now about twice as wide. The discretized exogenous stochastic process  $(y_t^T, r_t)$  is the same as before.

Under this calibration, we find a unique unregulated equilibrium. Figure 9 displays the equilibrium debt distribution in the unregulated and regulated economies. The fact that the former lies to the right of the latter implies that the unregulated economy displays overborrowing. The intuition of this result is that the absence of self-fulfilling crises removes one source of precautionary savings in the unregulated economy.

## 9 Conclusion

In open economy models in which borrowing is limited by the value of tradable and non-tradable output, the equilibrium value of collateral is increasing in end of period external debt. For plausible calibrations, this relationship can become perverse, in the sense that an increase in debt increases collateral by more than one for one. That is, as the economy chooses higher debt levels it becomes less leveraged. This problem can give rise to a non-convexity whereby two disjoint ranges of external debt for which the collateral constraint is satisfied are separated by a range for which the collateral constraint is violated.

This paper shows that in this environment, the economy displays self-fulfilling financial crises in which pessimistic views about the value of collateral induce agents to deleverage. In the context of a stochastic economy and under plausible calibrations, the paper shows that there exist equilibria with underborrowing, in the sense that the equilibrium level of debt is lower than what is optimal for a social planner with access to capital control taxes.

The underborrowing result stands in contrast to the overborrowing result stressed in the related literature. Underborrowing emerges in the present context because in economies that are prone to self-fulfilling financial crises, individual agents engage in excessive precautionary savings as a way to self insure.

The paper addresses the issue of implementation of the optimal equilibrium. This is a nontrivial problem, because the optimal policy only specifies what taxes are levied in equilibrium, but not what taxes would be levied off equilibrium. As a result, the optimal capital control policy does not ensure implementation of the optimal allocation. In particular, other, possibly welfare inferior, equilibria may be consistent with the optimal capital control policy. This paper shows that a capital control policy that threatens to tax capital outflows in

the event of a self-fulfilling financial crisis can make such events incompatible with a rational expectations equilibrium and therefore eliminate them as possible outcomes, ensuring the emergence of the desired equilibrium.

# Appendix

## A Proof of Proposition 1: Existence of Multiple Equilibria When $0 < \xi < 1$

Because the right-hand side of the period-0 collateral constraint (23) is strictly convex, it is clear that for any initial level of debt  $d_0 < \tilde{d}$ , a necessary and sufficient condition for the right-hand side of the period-0 collateral constraint to cross the 45-degree line to the left of  $d_0$  is that at the value of  $d$  at which the slope of the right-hand side of the period-0 collateral constraint takes the value of one, that is, the  $d$  such that  $S(d; d_0) = 1$ , the period-0 collateral constraint be violated. From equation (24), we obtain that the pairs  $(d, d_0)$  for which  $S(d; d_0) = 1$  satisfy

$$y^T + \frac{d}{1+r} - d_0 = \left[ \kappa \left( \frac{1-a}{a} \right) \frac{1}{(1+r)\xi} \right]^{\frac{\xi}{\xi-1}} \equiv \hat{c}_0^T > 0. \quad (40)$$

We then have that the set of values of  $d$  for which  $S(d; d_0) = 1$  and the collateral constraint is violated or binding is given by

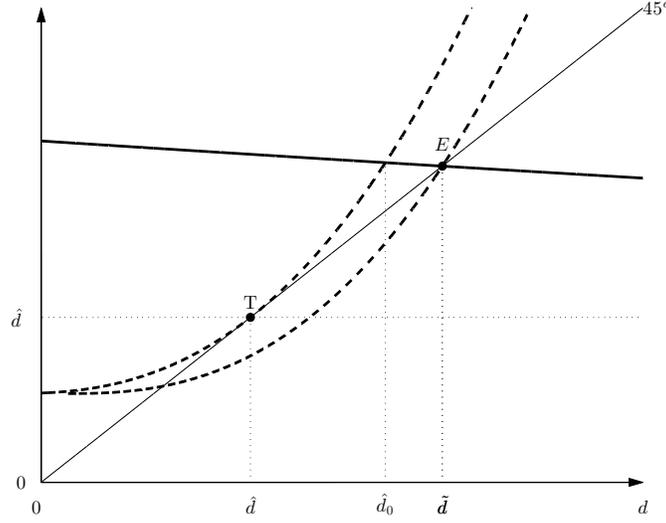
$$d \geq \kappa \left[ y^T + \frac{(1-a)}{a} (\hat{c}_0^T)^{\frac{1}{\xi}} \right] \equiv \hat{d} > 0.$$

Evaluating the period-0 resource constraint, equation (16), at  $c_0^T = \hat{c}_0^T$  and  $d_1 = \hat{d}$ , we have that the initial values of debt,  $d_0$ , for which the period-0 collateral constraint binds to the left of  $d_0$  are given by

$$d_0 \geq y^T + \frac{\hat{d}}{1+r} - \hat{c}_0^T \equiv \hat{d}_0. \quad (41)$$

Figure 10 depicts the right-hand side of the period-0 collateral constraint associated with  $d_0 = \hat{d}_0$  as a function of period-1 debt,  $d$ . This function is tangent to the 45-degree line at  $d = \hat{d}$  (point  $T$  in the figure), and intersects the steady-state collateral constraint at  $d = \hat{d}_0$ . Recalling that  $d_0$  cannot exceed  $\tilde{d}$  (since no steady-state equilibrium exists if  $d_0 > \tilde{d}$ ), we have that the above interval for  $d_0$  is meaningful only if  $\hat{d}_0 < \tilde{d}$ . But this is indeed the case. In subsection 4.1, we showed that multiple equilibria exist for values of  $d_0$  in a small neighborhood left of  $\tilde{d}$  if and only if  $S(\tilde{d}; \tilde{d}) > 1$ . This result and the fact that  $\hat{d}_0$  is the lower bound of the interval containing all values of  $d_0$  for which the period-0 collateral constraint binds for a value of  $d$  to the left of  $d_0$ , establish that  $\hat{d}_0 < \tilde{d}$  if and only if  $S(\tilde{d}; \tilde{d}) > 1$ , as we had set out to show.

Figure 10: Initial Debt Levels for which Multiple Equilibria Exist when  $0 < \xi < 1$



Notes. The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (21). The upward-sloping broken lines are the right-hand side of the period-0 collateral constraint, given in equation (23), for two values of  $d_0$ ,  $\hat{d}_0$  and  $\tilde{d}$ . Multiple equilibria exist for initial debt levels,  $d_0$ , in the interval  $[\hat{d}_0, \tilde{d}]$ .

## B Proof of Corollary 1: Existence of Two Self-Fulfilling Financial-Crisis Equilibria when $0 < \xi < 1$

As explained in subsection 4.2, for there to exist two self-fulfilling financial-crisis equilibria, the second intersection of the right-hand side of the period-0 collateral constraint with the 45-degree line must occur with a positive slope. This requirement ensures that in the second self-fulfilling financial-crisis equilibrium consumption of tradables is positive in period 0. Because  $\xi$  is less than one, the right-hand side of the period-0 collateral constraint is convex. Thus, the right-hand side of the period-0 collateral constraint will cross the 45-degree line with positive slope twice only if the period-0 collateral constraint is not violated at the level of debt,  $d$ , at which the right-hand side of the period-0 collateral constraint has a slope of zero. Formally, using equation (24), we have that the right-hand side of the period-0 collateral constraint has a slope of zero when

$$y^T + \frac{d}{1+r} - d_0 = 0. \quad (42)$$

Combining this expression with the period-0 collateral constraint given in equation (23), yields

$$d \leq \kappa y^T.$$

Now evaluating the period-0 resource constraint, equation (16), at  $c_0^T = 0$  and  $d_1 = \kappa y^T$ , we have that the initial values of debt,  $d_0$ , for which the period-0 collateral constraint is not violated when its right-hand side has slope 0 are given by

$$d_0 \leq \left(1 + \frac{\kappa}{1+r}\right) y^T. \quad (43)$$

This condition does not ensure two crossings of the right-hand side of the period-0 collateral constraint with the 45-degree line. It only ensures that for any  $d_0$  in this interval at the value of  $d$  at which the associated right-hand side of the period-0 collateral constraint has slope zero the period-0 collateral constraint is not violated. So it does not rule out that for some values of  $d_0$  in this interval the period-0 collateral constraint never bind. To rule out this situation, we must impose, in addition, the restriction  $d_0 \in [\hat{d}_0, \tilde{d}]$  which ensures the existence of one crossing with positive slope to the left of  $d_0$  (see Proposition 1). Thus we have that a necessary and sufficient condition for the existence of two self-fulfilling financial-crisis equilibria to exist is

$$d_0 \in \left[ \hat{d}_0, \min \left( \left(1 + \frac{\kappa}{1+r}\right) y^T, \tilde{d} \right) \right).$$

This interval is meaningful, because, comparing the expression for  $\hat{d}_0$  given in Proposition 1 with  $\left(1 + \frac{\kappa}{1+r}\right) y^T$ , we have that  $\left(1 + \frac{\kappa}{1+r}\right) y^T > \hat{d}_0$ . It can be shown that  $\left(1 + \frac{\kappa}{1+r}\right) y^T < \tilde{d}$  if and only if  $S(\tilde{d}; \tilde{d}) > 1/\xi$ .

## C Proof of Proposition 2: Existence of Multiple Equilibria When $\xi > 1$

When  $\xi > 1$  the right-hand side of the period-0 collateral constraint is increasing and concave in  $d$ . For  $d = d_0 \in (0, \tilde{d})$ , the right-hand side of the period-0 collateral constraint, equation (23), lies above the 45-degree line, that is, the collateral constraint is slack. Now, from the period-0 resource constraint (16) we have that, given a  $d_0 \in (0, \tilde{d})$ , the smallest value of  $d$  such that  $c_0^T$  is non-negative is given by  $(1+r)(d_0 - y^T)$ . If at  $d = (1+r)(d_0 - y^T)$ , the right-hand side of (23) lies below the 45-degree line, then there exists a value of  $d$  in the interval  $((1+r)(d_0 - y^T), d_0)$ , for which (23) holds with equality. At  $d = (1+r)(d_0 - y^T)$ , the right-hand side of (23) is equal to  $\kappa y^T$ . Thus we need that  $\kappa y^T < (1+r)(d_0 - y^T)$ . Rewrite this inequality as  $(1 + \kappa/(1+r))y^T < d_0$ . The question is then whether there exist any  $d_0$  that satisfy this inequality and also  $d_0 < \tilde{d}$ , that is, whether the interval  $\left((1 + \kappa/(1+r))y^T, \tilde{d}\right)$  is non-empty.

Combining the definition of  $\tilde{d}$  given in equation (22) with the resource constraint (16)

and the slope of the right-hand side of the period-0 collateral constraint given in (24) both evaluated at  $\tilde{d}$ , we can write

$$\left(1 + \frac{\kappa}{1+r}\right) y^T = \tilde{d} + \tilde{c}^T (1 - \xi \tilde{S}),$$

where  $\tilde{S} \equiv S(\tilde{d}; \tilde{d})$  and  $\tilde{c}^T \equiv y^T - \frac{r}{1+r} \tilde{d}$  is the level of tradable consumption in a steady-state with debt equal to  $\tilde{d}$ , which, as shown in section 3, is strictly positive. It follows that  $\left(1 + \frac{\kappa}{1+r}\right) y^T < \tilde{d}$  (i.e., the interval is non-empty) if and only if  $\tilde{S} > 1/\xi$ . It follows that a self-fulfilling financial-crisis equilibrium exists if and only if  $\tilde{S} > 1/\xi$ . We have also shown that if this condition is met, the range of initial values of debt for which self-fulfilling crises exist is given by  $d_0 \in \left(\left(1 + \frac{\kappa}{1+r}\right) y^T, \tilde{d}\right)$ .

## D Constraints of the Social-Planner's Problem

**Proposition 3** *Processes  $c_t^T$  and  $d_{t+1}$  satisfy equilibrium conditions (27)-(33) if and only if they satisfy (27) and (35).*

**Proof:** Suppose  $c_t^T$  and  $d_{t+1}$  satisfy (27) and (35). We must establish that (27)-(33) are also satisfied. Obviously (27) is satisfied. Now pick  $p_t$  to satisfy (31). This is possible, because the process  $c_t^T$  is given. Use this expression to eliminate  $p_t$  from (28). The resulting expression is (35), establishing that (28) holds. Next, pick  $\lambda_t$  to satisfy (29). Now, set  $\mu_t = 0$  for all  $t$ . It follows immediately that the slackness condition (32) and the non-negativity condition (33) are satisfied. Finally, pick  $\tau_t$  to ensure that (30) holds, that is, set

$$\tau_t = 1 - \beta(1+r_t) \mathbb{E}_t \frac{U'(A(c_{t+1}^T, y^N)) A_1(c_{t+1}^T, y^N)}{U'(A(c_t^T, y^N)) A_1(c_t^T, y^N)}. \quad (44)$$

Next, we need to show the reverse statement, that is, that processes  $c_t^T$  and  $d_{t+1}$  that satisfy (27)-(33) also satisfy (27) and (35). Obviously, (27) is satisfied, and combining (28) with (31) yields (35). This completes the proof of the equivalence of the constraint set (27)-(33) and the constraint set (27) and (35).

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