

INSTRUCTIONS: This examination is 3 hours. The exam consists of two parts. Choose a total of **seven problems**, including **at least three from each part**. Indicate on the first page of your exam the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

PART I

1. Prove that every group of order 15 is cyclic.
2. Consider the set of rational functions $X = \{f_1, f_2, f_3, f_4\}$, where

$$\begin{array}{ll} f_1(x) = x & f_2(x) = -x \\ f_3(x) = \frac{1}{x} & f_4(x) = -\frac{1}{x} \end{array}$$

- (a) Prove that X is a group with respect to the binary operation of composition of functions.
 - (b) Call the resulting group V . Compute its automorphism group. (The V is from the German word for four, which is *Vier*.)
3. A group G is *Hopfian* if it does not contain a normal subgroup $N \neq \{e\}$ such that $G/N \cong G$.
 - (a) Prove that every finite group is Hopfian.
 - (b) Prove that if G is Hopfian and $f : G \rightarrow G$ is a surjective homomorphism, then G is an automorphism.
 4. Determine the splitting field for the polynomial $t^8 - 3$ over \mathbb{Q} and compute a primitive element for this field.
 5. Compute the Galois groups of the following polynomials over \mathbb{Q} :
 - (a) $f(t) = t^3 + 1$
 - (b) $f(t) = t^3 - 2$
 6. Does the symmetric group on 3 letters embed inside $\text{GL}(2, \mathbb{C})$, the 2×2 group of invertible matrices with complex coefficients? Explain your answer.

PART II

1. Let p be a prime number. Let $T = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod{p}\}$.
 - (i) Show that T is a finitely generated free \mathbf{Z} -module by exhibiting a basis. More precisely, show that $T \cong \mathbf{Z} \times \mathbf{Z}$ as \mathbf{Z} -modules.
 - (ii) Show that $T \not\cong \mathbf{Z} \times \mathbf{Z}$ as \mathbf{Z} -algebras (where both rings are considered \mathbf{Z} -algebras via the diagonal embedding).
 - (iii) Show that $\mathfrak{m} := \{(pa, pb) \in T \mid a, b \in \mathbf{Z}\}$ is a maximal ideal of T . Prove or disprove the following statement: \mathfrak{m} is a principal ideal of T .

2. Let F be a field, $\alpha \in F$, $\alpha \neq 0$. Let M be an $F[x]$ -module which is finite-dimensional as an F -vector space. Let N be a one-dimensional F -vector space which is given an $F[x]$ -module structure by $x \cdot n := \alpha n$ for $n \in N$. Show that if $M \otimes_{F[x]} N \neq 0$, then M has an $F[x]$ -submodule isomorphic to $F[x]/(x - \alpha)^s$ for some $s \geq 1$.

3. Let $R = \mathbf{Z} \times \mathbf{Z}$. Show that $F := \mathbf{Q} \times \mathbf{Q}$ is the total ring of fractions of R and that R is integrally closed in F .

4. Show that $\mathbb{Z}[\sqrt{-11}]$ is not a Euclidean domain.

5. Let R be a commutative ring with identity. Let I be an ideal of R . Show that if R/I is a flat R -module then $I = I^2$.

6. (a) Let k be an algebraically closed field. Prove that every maximal ideal in the polynomial ring $R = k[x_1, \dots, x_n]$ has the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.
(b) Show that a family of polynomial functions on k^n with no common zeros generates the unit ideal of R .