CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SUMMER 2020

Instructions: The exam consists of two parts. Choose a total of *seven problems*, including at least three from each part. Indicate on the first page of your exam the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Notation: In this exam $\mathbf{Z}$ stands for the ring of rational integers, $\mathbf{Q}$ for the field of rational numbers and $\mathbf{R}$ for the field of real numbers.

Part 1

(1) Consider the field $K = \mathbf{Q}(\sqrt{3}, \sqrt{5})$.
   (i) Prove that K is a finite Galois extension of $\mathbf{Q}$ and compute the Galois group $\mathrm{Gal}(K/\mathbf{Q})$.
   (ii) Find all subgroups of $\mathrm{Gal}(K/\mathbf{Q})$ and the corresponding fixed fields.

(2) A group G is *locally finite* if every finitely generated subgroup of G is finite.
   (i) Prove that every locally finite group is a torsion group.
   (ii) Prove that if G is locally finite and N is a normal subgroup of G, then the quotient group G/N is locally finite.

(3) Let F be a field and let $F(z)$ be the field of rational functions with coefficients in F. Let $\varphi : F(z) \to F(z)$ be a ring homomorphism such that φ is the identity on F, that is, $\varphi(a) = a$ for all $a \in F$.
   (i) Prove that φ is injective, that φ is determined by the rational function $w = \varphi(z) \in F(z)$, and that the image of φ is the field

$$\varphi(F(z)) = F(w) \cong F(z).$$

   (ii) Let $w = f(z)/g(z)$ be a nonconstant rational function in $F(z)$, where $f(z)$ and $g(z)$ are relatively prime polynomials in $F[z]$. Define the ring homomorphism $\varphi : F(z) \to F(z)$ by $\varphi(z) = w$. Consider the polynomial

$$h(t) = f(t) - g(t)w \in F(w)[t].$$

   Prove that $h(z) = 0$.

(4) Let $n \geq 2$ be a positive integer and write S_n for the symmetric group on n letters. Let N be a normal subgroup of S_n. Prove that if N contains a transposition then $N = S_n$.

(5) Let G be a group that acts on a finite set X, and let k be the number of orbits of the action of G on X. A point $x \in X$ is a fixed point of an element $g \in G$ if $gx = x$. Let $\mathrm{FP}(g) = \{x \in X \mid gx = x\}$ be the set of fixed points of g. Prove that the number of orbits is the average number of fixed points, that is,

$$k|G| = \sum_{g \in G} |\mathrm{FP}(g)|.$$

(6) Compute the splitting field E of the polynomial $t^5 - 3$ over $\mathbf{Q}$, and the Galois group $\mathrm{Gal}(E/\mathbf{Q})$. Find the minimal polynomial over $\mathbf{Q}$ of a primitive element of E. How many fields F exist such that $\mathbf{Q} \subset F \subset E$ and $[F : \mathbf{Q}] = 5$?

PART 2

(1) Consider the ring $\mathbf{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbf{Z}\}$.
 (i) Show that $\mathbf{Z}[\sqrt{5}]$ is not integrally closed in its field of fractions.
 (ii) Show that $\mathbf{Z}[\sqrt{5}] \otimes_{\mathbf{Z}} \mathbf{R}$ is not an integral domain.

(2) Let $R = \{(f, g, h) \mid f, g, h \in \mathbf{Q}[[X]], f \equiv g \equiv h \pmod{X}\}$. Find $\dim_{\mathbf{Q}} \mathrm{Hom}_{\mathbf{Q}[[X]]}(R, \mathbf{Q})$. Here $\mathbf{Q}[[X]]$ denotes the ring of formal power series with coefficients in $\mathbf{Q}$ and we treat $\mathbf{Q}$ as $\mathbf{Q}[[X]]$-module via the map $X \mapsto 0$.

(3) Give an example of a projective module which is not free. If M and N are projective modules over a commutative ring R with 1, show that $M \oplus N$ is also projective.

(4) Let R be a Noetherian ring. Let $\phi : R \to R$ be a surjective ring homomorphism. Show that ϕ is an isomorphism.

(5) Let $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and let I be the identity matrix in $\mathrm{GL}_4(\mathbf{Q})$.
 (i) Show that the characteristic polynomial of M is $(x-1)^4$.
 (ii) Find the dimension of the eigenspace $V_1 = \{v \in \mathbf{Q}^4 \mid Mv = v\}$.
 (iii) Find the minimal polynomial of M.
 (iv) Find the Jordan canonical form of M.

(6) Let R be a commutative ring with identity. Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{m} \neq \mathfrak{m}^2$. Set $k := R/\mathfrak{m}$.
 (i) Show that $\mathfrak{m}/\mathfrak{m}^2$ is naturally a k-vector space.
 (ii) Show that if $\mathfrak{m}$ is principal then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.
 (iii) Show by example that in general one can have $\dim_k \mathfrak{m}/\mathfrak{m}^2 > 1$.