CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
Summer 2021
3 hours

Instructions. The exam consists of two parts. Choose a *total of six problems*, including *three from each part*. Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Find the order of the Galois group of the polynomial $f(x) = x^5 - 2$ over $\mathbb{Q}$.
2. Let $G = SL(3, \mathbb{Z})$ and let
$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$
Prove that H is a subgroup of G with $[G : H] = \infty$.
3. Let $G = SL(2, \mathbb{Z})$ and let $X = \mathbb{Z}^2$. Consider the action $G \times X \to X$
$$\left(A, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \mapsto A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
where $A \in SL(2, \mathbb{Z})$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{Z}^2$. You can take for granted that this is a group action.
   a. For the vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ prove that its orbit is equal to
$$O_{e_1} = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} | m, n \in \mathbb{Z}, \gcd(m, n) = 1 \right\}$$
   b. Compute $\mathrm{Stab}_G(e_1)$ and $[G : \mathrm{Stab}_G(e_1)]$.
4. a. Give an example of a non-commutative ring R and a subset $I \subseteq R$ such that I is a left ideal in R but not a right ideal in R. Justify that your example has the required properties.
   b. Give an example of a Unique Factorization Domain which is not a Principal Ideal Domain. Justify that your example has the required properties.
   c. Given an example of a unital commutative ring R and ideals I, J in R such that $I \cdot J \neq I \cap J$. Justify that your example has the required properties.
5. Prove that there are no simple groups G of order $5103 = 3^6 \cdot 7$. Hint: Consider the action of G on the set of 3-Sylow subgroups by conjugation; what can you say about its kernel?
6. a. Prove that the number $\alpha = \frac{\sqrt{2}+1}{3\sqrt{3} - \sqrt[8]{5} + 10}$ is constructible over $\mathbb{Q}$.
   b. Let β be a real root of $x^3 - 20x^2 + 91x - 63 = 0$. Prove that β is not constructible over $\mathbb{Q}$.

Part II

7. Let R be a unital ring. An element $e \in R$ is an idempotent if $e^2 = e$.
   a. Prove that if e is an idempotent, then $R \cong Re \oplus R(1 - e)$ as left R-modules.
   b. Prove that Re is a projective module.
8. Let $n \geq 2$. Let R be the ring $\mathbb{Z}/n\mathbb{Z}$.
   a. Show that every ideal of R is of the form $(d + n\mathbb{Z})$ where $d \mid n$.

b. Show that if $f\colon (d+n\mathbb{Z}) \to R$ is a left R-module homomorphism, then $f(d+n\mathbb{Z}) \in (d+n\mathbb{Z})$. Hint: use that a cyclic group of order n has a unique subgroup of order d for any divisor d of n and if $d_1 \mid d_2 \mid n$, then the subgroup of order d_1 is contained in the subgroup of order d_2.

c. Deduce that f extends to left R-module homomorphism $F\colon R \to R$. Hint: if $f(d+n\mathbb{Z}) = dk + n\mathbb{Z}$, what should $F(1+n\mathbb{Z})$ be?

d. Use Baer's criterion to explain why R is an injective left R-module.

9. a. Give one (and only one) representative from each conjugacy class of 5×5 matrices over $\mathbb{C}$ with characteristic polynomial $(x-2)(x-3)^4$. Justify your work.

   b. Find the rational canonical form over $\mathbb{Q}$ of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and its minimal polynomial.

10. Let R be a commutative ring with 1 and I an ideal.

   a. Prove that I annihilates I/I^2.

   b. Give an example of a commutative ring R with 1 and an ideal I of R so that $I/I^2 \otimes_R I$ is a nonzero module.

11. Let M be a left R-module. Then M is Noetherian if whenever

$$M_1 \subseteq M_2 \subseteq \cdots$$

is an ascending chain of submodules, there is some index $N > 0$ such that $M_n = M_N$ for all $n \geq N$.

   a. Show that a left R-module M is Noetherian if and only if each submodule of M is finitely generated.

   b. Show that if M is Noetherian, then any surjective R-module homomorphism $f\colon M \to M$ is an isomorphism. Hint: Look at the ascending chain

$$\ker f \subseteq \ker f^2 \subseteq \cdots$$

   and suppose $\ker f^n = \ker f^{n+1}$. Use that f^n is surjective to show if $m \in \ker f$, then $m = 0$.

12. Let G be a finite group

   a. Let N be a normal subgroup of G. Suppose that $\pi\colon G \to G/N$ is the projection and let $\rho\colon G/N \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Prove that $\rho \circ \pi\colon G \to \mathrm{GL}_n(\mathbb{C})$ is an irreducible representation.

   b. χ be the character of a nontrivial irreducible representation of G (over $\mathbb{C}$) (recall that the trivial representation of G is the one-dimensional representation sending each element of G to 1). Prove that $\sum_{g \in G} \chi(g) = 0$.