CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2015
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part*. Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. a. Construct a non-abelian group G of order 55.
   b. Show that G is unique up to isomorphism.

2. Prove that there does not exist a simple group of order 380.

3. Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and let B be the subgroup of A generated by $u = (6, 0, 6)$, $v = (0, 3, 12)$ and $w = (6, -3, 0)$. Write A/B as a direct sum of cyclic groups.

4. Factor each of the following polynomials into irreducibles in the indicated polynomial ring and verify irreducibility of the factors.
   a. $f(x) = 2x^4 - x^3 - 10x^2 - x + 3$ in $\mathbb{Z}[x]$.
   b. $g(x) = x^6 + (2i - 1)x^3 - (i + 2)$ in $\mathbb{Z}[i][x]$, where $\mathbb{Z}[i]$ is the ring of Gaussian integers.
   c. $h(x) = x^5 + x + 1$ in $\mathbb{F}_2[x]$, where $\mathbb{F}_2$ is the field with 2 elements.

5. In $R = \mathbb{Z}[\sqrt{-17}]$, we have $(2 + \sqrt{-17}) \cdot (2 - \sqrt{-17}) = 3 \cdot 7$. Find prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ in R such that $\mathfrak{p}_1\mathfrak{p}_2 = (2 + \sqrt{-17})R$. Justify that the ideals you found are prime and that their product is the principal ideal $(2 + \sqrt{-17})R$.

6. Let $Z(G)$ denote the center of a finite group G and suppose that $[G : Z(G)] = n > 1$. Prove that each conjugacy class of G has strictly fewer than n elements.

Part II

7. Suppose that an integral domain D contains the field F and that D is integral over F, i.e. every element of D satisfies a *monic* polynomial with coefficients in F. Prove that D is a field.

8. For $a \in \mathbb{Z}$, consider $f(x) = x^4 + ax + 1$ over $F = \mathbb{Q}$ and the reduction of f over finite fields $F = \mathbb{F}_p$ with p prime. Describe the Galois group $\mathrm{Gal}_F(f)$ of a splitting field of f over F in the following cases:
   a. $\mathrm{Gal}_{\mathbb{F}_2}(f)$ when a is odd;
   b. $\mathrm{Gal}_{\mathbb{F}_3}(f)$ when $a \equiv 1 \pmod 3$;
   c. $\mathrm{Gal}_{\mathbb{Q}}(f)$ when $a \equiv 1 \pmod 6$.

9. Let F be field of characteristic 0. For p prime, let K be the splitting field of $x^p - 1$ over F.
   a. Sketch the proof that $\mathrm{Gal}(K/F)$ is isomorphic to a subgroup of the multiplicative group $(\mathbb{Z}/p)^\times$ of units modulo p.
   b. Fix an integer $d > 1$. According to Dirichlet's theorem, there are infinitely many primes p such that $p \equiv 1 \pmod d$. Prove that there is a Galois extension $K/\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order d.

– over –

10. Let R be a commutative ring and let D be a unital module.

a. Prove that for unital R-modules M_1, M_2 there is an isomorphism:
$$D \otimes_R (M_1 \oplus M_2) \simeq (D \otimes_R M_1) \oplus (D \otimes_R M_2).$$

b. Let I be a (possibly infinite) indexing set and let $\{M_i\}_{i \in I}$ be a collection of unital R-modules indexed by the elements of I. Give an example to show that $D \otimes_R \left(\prod_{i \in I} M_i\right)$ is not necessarily isomorphic to $\prod_{i \in I}(D \otimes M_i)$, i.e., that tensor product does not necessarily commute with (arbitrary) direct products.

11. Over the finite field $\mathbb{F}_{13}$, we have the factorization into irreducible polynomials:
$$x^7 - 1 = (x-1)(x^2+3x+1)(x^2+5x+1)(x^2+6x+1).$$
Write $G = \mathrm{GL}_2(\mathbb{F}_{13})$ for the multiplicative group of 2×2 invertible matrices over $\mathbb{F}_{13}$ and let g be an element of order 7 in G.

a. Justify the following claim: g can be diagonalized over the field $\mathbb{F}_{169}$ but not over $\mathbb{F}_{13}$.

b. Find representatives for the distinct conjugacy classes of matrices of order 7 in $\mathrm{GL}_2(\mathbb{F}_{13})$.

12. Let k be a field of characteristic 0 and let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. Recall that $\mathbb{A}^n = k^n$ denotes affine n-space over k and that the zero-set of an ideal I of R is $\mathcal{Z}(I) = \{\mathbf{a} \in \mathbb{A}^n \mid f(\mathbf{a}) = 0 \text{ for all } f \in I\}$.

a. For ideals I and J of R, prove that $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(IJ)$.

b. Let $I = (y^2 - xz - z^2, x^2 - 3xz)$ in $k[x,y,z]$. Find prime ideals P_j in $k[x,y,z]$ such that
$$\mathcal{Z}(I) = \mathcal{Z}(P_1) \cup \mathcal{Z}(P_2) \cup \ldots \cup \mathcal{Z}(P_m).$$