CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2018
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part*. Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let $\mathbb{F}_2$ be the field with two elements. For $f(x) = x^4 + x^3 + 1 \in \mathbb{F}_2[x]$, let $E = \mathbb{F}_2[x]/(f)$.
   (a) Show that E is a field and find $|E|$.
   (b) How many generators for the cyclic group $E^\times$ are there? Find one (in coset notation).
   (c) Find the inverse of $x^3 + 1 + (f)$ in E (in coset notation).
   (d) Does E contain a subfield with exactly 8 elements?
2. Let k be a field. Let $X = k \cup \{\infty\}$. You may assume that $SL_2(k)$ (the group of determinant one 2×2 matrices over k) acts on X by fractional linear transformations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$
   for $z \in X$. (Here, we adopt the "obvious" conventions regarding arithmetic with ∞.)
   (a) Let $\mathbb{F}_2$ be the field with two elements. Show that $K = \mathbb{F}_2[x]/(x^2 + x + 1)$ is a finite field with 4 elements.
   (b) Prove that $SL_2(K)$ is a simple group. (**Hint:** Use the action of $SL_2(K)$ by fractional linear transformations on the five element set $K \cup \{\infty\}$.)
3. Let R be a commutative ring with identity and let S be a multiplicative subset of R. Prove that, for each R-module M, the $S^{-1}R$-modules $S^{-1}R \otimes_R M$ and $S^{-1}M$ are isomorphic.
4. Let R be a commutative ring with identity.
   (a) Give the definition of a Noetherian module over the ring R.
   (b) Give an example of a commutative ring with identity R and a finitely generated R-module that is not Noetherian.
5. Let $p < q < r$ be three distinct primes. Prove that a group of order pqr is not simple.
6. Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and let B be the subgroup of A generated by $u = (0, 10, 10)$, $v = (5, 20, 0)$ and $w = (-5, 0, 10)$. Write A/B as a direct sum of cyclic groups.

Part II

7. Show that the primitive 12^{th} roots of unity of the 12^{th} cyclotomic field form a normal basis over the rationals.
8. Let p be a prime and $A, B \in SL_2(\mathbb{F}_p)$ with trace(A) = trace(B) (where $SL_2(\mathbb{F}_p)$ is the group of 2×2 matrices over the p-element field with determinant one).
   (a) Must A and B be conjugate in $SL_2(\mathbb{F}_p)$?
   (b) Is it ever possible for there to exist a representation of $SL_2(\mathbb{F}_p)$ where the images of A and B have different traces?
9. (a) How many intermediate extensions F exist with $\mathbb{Q} \leq F \leq \mathbb{Q}[\sqrt{2}, \sqrt{3}]$?
   (b) Show that $\mathbb{F}_3(X, Y)$ is not a simple extension of $\mathbb{F}_3(X^3, Y^3)$ where $\mathbb{F}_3$ is the field of three elements. Recall that an extension is simple if it is generated by a single element. (**Hint:** Show that $\mathbb{F}_3(X, Y)$ is a degree 9 extension of $\mathbb{F}_3(X^3, Y^3)$ but each of its elements has degree at most 3.)
10. Let $f(x) = x^4 - x^2 - 1$.

(a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
(b) Find a splitting field $K \subseteq \mathbb{C}$ for f.
(c) Prove that the Galois group of f is isomorphic to the dihedral group of order 8.

11. Let K be a field and A a finite dimensional commutative K-algebra with identity.
    (a) Prove that every non-zero-divisor of A is a unit.
    (b) Prove that every prime ideal $\mathfrak{p}$ of A is maximal. (**Hint:** apply (a) to $A/\mathfrak{p}$.)
12. Let G be a finite p-group with p-prime. Let $\mathbb{F}_p$ be the p-element field.
    (a) Let $\rho\colon G \to GL_n(\mathbb{F}_p)$ be a representation. Prove that there is a non-zero vector $v \in \mathbb{F}_p^n$ that is fixed by all elements of G. (**Hint:** count orbit sizes or use Sylow's theorem.)
    (b) Prove that the trivial representation is the only irreducible representation of G over $\mathbb{F}_p$.

Reminder: please indicate on the cover of your answer book the seven problems you have chosen for grading.