Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
December 2020

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\Delta^* := \Delta \setminus \{0\}$
- $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$: the right half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

Part A. Answer any two of the following three questions.

A1. State and prove the Schwarz lemma.

A2. If f is a meromorphic function on an open set G (in $\mathbb{C}$), show that there are holomorphic functions g and h on G such that $f = g/h$.

A3. State Jensen's formula and the Schwarz's reflection principle.

Part B. Solve any two of the following four problems.

B1. Let G be a region with the property that every harmonic function in G has a harmonic conjugate in G. Prove that if f is holomorphic in G such that $f(z) \neq 0$ for all z in G, then there is a holomorphic function g in G such that $f(z) = e^{g(z)}$ for all z in G.

B2. Let $f : \Delta \to \mathcal{H}$ be a holomorphic map such that $f(0) = 1$ Prove the following:

(i) $\dfrac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \dfrac{1 + |z|}{1 - |z|}$ for all $z \in \Delta$.

(ii) $|f'(0| \leq 2$.

B3. Let R and R' be two rectangles in $\mathbb{C}$. Prove that any conformal map $f : R \to R'$ which maps vertices to vertices is of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

B4. Let f be an entire function that satisfies

$$|f(z)| \leq 2 \log(1 + |z|).$$

Prove that f is a constant.

PART C. SOLVE ANY FOUR OF THE FOLLOWING SIX PROBLEMS.

C1. Let $\lambda > 1$. Show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in $\mathcal{H}$. Show also that this solution must be real.

C2. Let f be an entire function, such that $f(z + w) = f(z)$ for some $w \in \mathbb{C}$, $w \neq 0$. Prove that there must exist some z_0 in $\mathbb{C}$ such that $f(z_0) = z_0$.

C3. Let $f : \Delta \to \Delta^*$ be holomorphic such that $f(0) = 1/2$. Show that $|f(1/2)| \geq 1/8$.

C4. Find Green's function for the strip $\{z : -\pi < \text{Im}(z) < \pi\}$ with the pole at $z = 0$.

C5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and $A = \{z : 1 < |z| < 2\}$. Given $s \in \mathbb{R}$, define a holomorphic function $f_s(z) := f(sz)$ with the domain A. Prove that the family $\{f_s\}_{s \in \mathbb{R}}$ is normal if and only if f is a polynomial.

C6. The series

$$f(z) := \sum_{n=1}^{\infty} z^{n!}$$

has radius of convergence 1. Prove that it cannot be analytically continued in a neighborhood of any point z with $|z| = 1$.