Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
September, 2015

<u>**Notations**</u>

- $\mathrm{Re}(z)$: the real part of a complex number z
- $\mathrm{Im}(z)$: the imaginary part of a complex number z
- $\mathbb{C}$: the complex plane
- $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$: the Riemann sphere
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω

PART I: Do Any TWO Problems. Do NOT do more than two.

1. Let V be a nonempty proper region in $\mathbb{C}$ (which means, $V \neq \mathbb{C}$) with the following property: every $g \in \mathcal{O}(V)$ that omits the value zero has a square root in $\mathcal{O}(V)$. Show that there exists an $f \in \mathcal{O}(V)$ that maps V homeomorphically onto Δ.

2. Suppose $f \in \mathrm{Aut}(\Delta)$ such that f has two fixed points in Δ. Show that f must be the identity.

3. (a) State the maximum modulus principle for holomorphic functions. (b) State and prove Schwarz's lemma.

4. Show that any function that is meromorphic on $\widehat{\mathbb{C}}$ is rational.

PART II: Do Any TWO Problems. Do NOT do more than two.

1. Give precise statements of the following: (i) Little Picard Theorem (ii) Great Picard Theorem (iii) Schwarz's Reflection Principle (iv) Runge's Theorem

2. Let G be a region in $\mathbb{C}$, and let M be a fixed positive constant. Let $\mathcal{F}$ be the family of all functions $f \in \mathcal{O}(G)$ such that

$$\iint_G |f(z)|^2 dxdy \leq M.$$

Show that $\mathcal{F}$ is a normal family.

3. If $e^{f(z)} + e^{g(z)} = 1$ for all z, where f and g are entire functions, show that f and g are constants.

4. Let Ω be the open domain bounded by the two circles, $C_1 = \{z : |z-1| = 1\}$ and $C_2 = \{z : |z| = 2\}$. Find an explicit conformal mapping from Ω to Δ.

PART III: Do Any FOUR Problems. Do NOT do more than four.

1. Let G be a region in $\mathbb{C}$, such that every harmonic function in G has a harmonic conjugate in G. Show that every non-zero holomorphic function in G must have a holomorphic logarithm in G.

2. (a) Let f be an entire function such that $\mathrm{Re}(f(z)) \to 0$ as $|z| \to \infty$. Show that f is identically zero.

(b) Let D be bounded region in $\mathbb{C}$. Let $f : \overline{D} \to \mathbb{C}$ be a non-constant continuous function, which is holomorphic in D and satisfies $|f(z)| = 1$ for all $z \in \partial D$. Show that $f(z_0) = 0$ for some $z_0 \in D$. (Here, $\overline{D}$ denotes the closure of D, and ∂D denotes the boundary of D.)

3. How many zeroes does the equation

$$z^5 + z^4 + 6z + 1 = 0$$

have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$?

4. Evaluate the following definite integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

5. (a) Prove that $\Pi_{n\neq 0}\left(1 - \frac{z}{n}\right)e^{z/n}$ converges uniformly to an entire function. (b) Prove that $\frac{\sin \pi z}{\pi z} = \Pi_1^\infty\left(1 - \frac{z^2}{n^2}\right)$.

6. Prove that if u is a harmonic function on the domain $|z| < 1$, then for every $0 < r < 1$ and $|z| < r$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta.$$

7. Let Ω be a connected open set in the complex plane. Suppose $\{f_n\}$ is a sequence of injective functions in $\mathcal{O}(\Omega)$ which uniformly converge on compact subsets to a function f in $\mathcal{O}(\Omega)$. Show that f is also injective unless it is a constant.