Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Fall 2019

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\operatorname{Re}(z), \operatorname{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$: the upper half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

Part A. Answer any two of the following three questions.

A1. Describe all conformal self-maps of the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im}(z) < 1\}$.

A2. State a version of Montel's normality theorem that you are familiar with. Explain why every family of holomorphic functions $\mathbb{C} \to \mathbb{C} \smallsetminus \{0, 1\}$ must be normal. Then give an example of a family of holomorphic functions $\mathbb{C} \to \mathbb{C} \smallsetminus \{0\}$ that fails to be normal.

A3. State and prove the Schwarz lemma.

Part B. Solve any two of the following four problems.

B1. Let $f : \Delta \smallsetminus \{0\} \to \mathbb{C}$ be holomorphic such that $|f(z)| \leq -\log |z|$ in $\Delta \smallsetminus \{0\}$. Prove that $f(z) = 0$ for all z.

B2. Find all entire functions f such that $f(0) = 1$ and $|f(z)| \leq e^{3 \operatorname{Re}(z)}$ for all $z \in \mathbb{C}$.

B3. Let $f : \Delta \to \mathcal{U}$ be holomorphic, with $f(0) = i$. Prove the following:

(i) $\dfrac{1-|z|}{1+|z|} \leq |f(z)| \leq \dfrac{1+|z|}{1-|z|}$ for all $z \in \Delta$.

(ii) $|f'(0)| \leq 2$.

B4. Let $G \subset \mathbb{C}$ be a simply connected region. If $z_1, z_2 \in G$, prove that there exists a conformal map $f : G \to G$ which satisfies $f(z_1) = z_2$ and $f'(z_1) > 0$, and that such f is unique if $G \neq \mathbb{C}$.

PART C. SOLVE ANY FOUR OF THE FOLLOWING SIX PROBLEMS.

C1. Let h be a positive harmonic function in Δ. Show that there are harmonic functions $u, v : \Delta \to \mathbb{R}$ such that $h = e^u \sin v$.

C2. Let f be an entire function that is purely imaginary on the real axis and real on the imaginary axis. Prove that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

C3. Let f be an entire function that satisfies

$$\int_0^{2\pi} |f(re^{it})| \, dt \leq r^{5/2}$$

for all $r > 0$. Show that f is identically zero.

C4. Show that the function $f(z) = \sin^2 z - 9z^2$ has a holomorphic square root in the unit disk. In other words, there is a holomorphic function $g : \Delta \to \mathbb{C}$ such that $(g(z))^2 = f(z)$ for all $z \in \Delta$.

C5. Determine the number of zeros of the function $e^{z^2} - 4z^2$ in Δ.

C6. Show that there exists a sequence $\{p_n\}$ of polynomials such that $p_n(0) = 1$ for $n = 1, 2, 3, \ldots$ but $\lim_{n\to\infty} p_n(z) = 0$ for every $z \neq 0$.