Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Fall 2021

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded*. You have 3 hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\text{Re}(z), \text{Im}(z)$: The real and imaginary parts of a complex number z
- $\mathbb{C}$: The complex plane
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$: the upper half-plane
- By a “region” we mean a non-empty connected open set in $\mathbb{C}$
- By a “conformal map” $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

Part A. Answer any two of the following three questions.

A1. State and prove Hurwitz’s theorem.

A2. Give precise statements of the following: (i) The Argument principle for meromorphic functions (ii) The Cauchy integral formula for holomorphic functions (iii) Jensen’s formula.

A3. Give precise statements of the following: (i) the Monodromy theorem (ii) the little Picard theorem (iii) the Schwarz Lemma.

Part B. Solve any one of the following two problems.

B1. Let $f : \mathcal{U} \to \mathcal{U}$ be a holomorphic function. Prove that

$$\frac{|f'(z)|}{\text{Im } f(z)} \leq \frac{1}{\text{Im } z}$$

for all $z \in \mathcal{U}$.

B2. Let $f(z)$ be an entire function that satisfies

$$|f(z)| \le |z|^{10} \log(1 + |z|).$$

Prove that $f(z)$ is a polynomial of degree at most 10.

Part C. Solve any four of the following six problems.

C1. Suppose Ω is a region, and D is a disc, such that $\overline{D} \subset \Omega$. If f is a nonconstant holomorphic function in Ω, and if $|f|$ is constant on ∂D (the boundary of D), prove that f has at least one zero in D.

C2. Find the number of zeros of the polynomial $p(z) = z^3 - 5z + 1$ in the annulus $\{z : \frac{1}{3} < |z| < 3\}$.

C3. Let $u(x, y)$ be a harmonic function on a simply connected region V. Prove that $u(x, y)$ must have a harmonic conjugate in V.

C4. Let f be an entire function and n a positive integer. Show that there exists an entire function g such that $g^n = f$ if and only if the orders of the zeros of f are divisible by n.

C5. Let $f : \{z : |z| < 2\} \to \{z : |z| < 5\}$ be a holomorphic function with $f(0) = i$. Prove that $f(z)$ does not have a zero in the set $\{z : |z| \le \frac{2}{5}\}$.

C6. a) Let $u(z)$ and $[u(z)]^2$ be real-valued harmonic functions on a domain Ω. Prove that $u(z)$ is constant on Ω.

b) Let $u(z)$ be a real-valued harmonic function on a domain Ω. Prove that any partial derivative of $u(z)$ is also a harmonic function.

c) Let $u : \{z : |z| < \frac{1}{2}\} \to \mathbb{R}$ be a positive harmonic function. If $u(0) = 1$ prove that

$$\frac{1}{3} \le u(-\frac{i}{4}) \le 3.$$