Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
May, 2015

<u>Notations</u>

- $\mathrm{Re}(z)$: the real part of a complex number z
- $\mathrm{Im}(z)$: the imaginary part of a complex number z
- $\mathbb{C}$: the complex plane
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$
- $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$: the Riemann sphere
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\Delta^* := \Delta \setminus \{0\}$
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω

PART I: Do Any TWO Problems. Do NOT do more than two.

1. Show that each element of $\mathrm{Aut}(\Delta)$ can be expressed as

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}},$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$.

2. Give precise statements of the following: (a) the *Riemann mapping theorem*, (b) the *Mittag-Leffler theorem*, (c) the *Weierstrass theorem* (for zeroes of holomorphic functions in an open set W in $\widehat{\mathbb{C}}$, $W \neq \widehat{\mathbb{C}}$), (d) the *Monodromy theorem*.

3 Describe the following: (a) $\mathrm{Aut}(\mathbb{C}^*)$, (b) $\mathrm{Aut}(\Delta^*)$. (You do not need to prove your claims.)

4. (a) Give a precise statement of the *Argument Principle*. (b) State *Morera's Theorem.*

PART II: Do Any TWO Problems. Do NOT do more than two.

1. Suppose $f \in \mathcal{O}(\Delta)$, and the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \Delta)$$

has radius of convergence 1. Show that f has at least one singular point on the unit circle.

2. Let f be a holomorphic function defined on $|z| < 1$ satisfying $|f(z)| < 1$. If $f(\frac{1}{2}) = \frac{1}{3}$, find a sharp upper bound for $|f'(\frac{1}{2})|$.

3. Let γ be the square with side length L and center point z_0. Find positive lower and upper bounds which are independent of L for the following integral.

$$\int_\gamma \frac{|dz|}{|z - z_0|}$$

4. Let f be an entire function with $|f(z)| \leq a|z|^b + c$ for all $z \in \mathbb{C}$ where a, b, c are all positive constants. Prove that f is a polynomial of degree at most b.

PART III: Do Any FOUR Problems. Do NOT do more than four.

1. Let $\mathbb{H}$ denote the right half-plane $\{z : \text{Re}(z) \geq 0\}$. If $f : \mathbb{H} \to \mathbb{H}$ is holomorphic and $f(1) = 1$ show that (i) $|f'(1)| \leq 1$ and (ii) $\left|\frac{f(z)-1}{f(z)+1}\right| \leq \left|\frac{z-1}{z+1}\right|$ for all z in $\mathbb{H}$.

2. (i) Suppose Ω is a simply connected region, and $u(x, y)$ is a harmonic function in Ω. Show that u must have a harmonic conjugate in Ω.

(ii) If $f \in \mathcal{O}(G)$, where G is any open set in $\mathbb{C}$, and if f has no zero in G, show that $\log|f|$ is harmonic in G.

3. Show that, for every real number $\lambda > 1$, the function

$$f(z) = ze^{\lambda - z} - 1$$

has exactly one zero in Δ and that it is real and positive.

4. Evaluate the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} dx$$

5. (a) Prove that $S = \sum_{n \neq 0} \frac{z}{n(z-n)}$ (the sum is over non-zero integers) converges absolutely and uniformly on all compact subsets of $\mathbb{C} - \{\text{non-zero integers}\}$. (b) Prove that $S + \frac{1}{z} = \pi \cot \pi z$.

6. (a) Define an elliptic function. (b) Prove that an elliptic function must have poles of order at least 2 (counted with multiplicity).

7 Suppose that $\phi(\zeta)$ is continuous on the arc γ. Consider the function,

$$F(z) = \int_\gamma \frac{\phi(\zeta) d\zeta}{\zeta - z}$$

Using the definition of the derivative show that $F(z)$ is analytic in each of the regions determined by γ and determine $F'(z)$.