Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
May, 2016

Notations

- $\mathrm{Re}(z)$: the real part of a complex number z
- $\mathrm{Im}(z)$: the imaginary part of a complex number z
- $\mathbb{C}$: the complex plane
- $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: the Riemann sphere
- $\Delta = \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane

PART I: Answer any TWO questions.

1. State and prove *Picard's little theorem.*

2. Give precise statements of the following: (i) the *Riemann mapping theorem*; (ii) the *Mittag-Leffler theorem*; (iii) the *Weierstrass theorem* (on zeros of a holomorphic function in a region); (iv) *Runge's theorem*; (v) the *Monodromy theorem.*

3. Suppose $\{u_n\}$ is a sequence of harmonic functions in a region $G \subset \mathbb{C}$ such that $u_1 \leq u_2 \leq u_3 \leq \cdots$. Show that $\{u_n\}$ converges to a harmonic function or to $+\infty$, uniformly on compact subsets of G.

4. (i) Show that the sum of the residues of an elliptic function is zero.

(ii) Show that a non-constant elliptic function has as many poles as zeros, counting multiplicity.

PART II: Answer any TWO questions.

1. Let f be holomorphic in Δ and $|f(z)| \leq 1$. If $z_0 \in \Delta$, show that

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} \leq \frac{2}{1 - |z_0||z|}$$

for all $z \neq z_0$.

2. Let $f(z)$ be holomorphic in Δ, $\mathrm{Re} f(z) > 0$, and $f(0) > 0$. Show that

$$f(0)\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq f(0)\frac{1 + |z|}{1 - |z|}$$

for all z in Δ.

3. If $f \in \mathrm{Aut}(\mathcal{U})$, show that

$$f(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are real and $ad - bc = 1$.

PART III: Answer any FOUR questions.

1. How many zeros does the equation

$$z^5 + iz^3 - 4z + i = 0$$

have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$?

2. Evaluate the following definite integral

$$\int_0^\infty \frac{dx}{1+x^6}.$$

3. Prove that every holomorphic function on $\widehat{\mathbb{C}}$ whose singularities are poles is a rational function. That is, it has the form $P(z)/Q(z)$, where P and Q are polynomials.

4. If $z_n = i/n$, show that the product $\prod_{n=1}^{\infty}(1 + z_n)$ diverges while the product $\prod_{n=1}^{\infty} |1 + z_n|$ converges.

5. Suppose p is an isolated singularity of a holomorphic function f and $\mathrm{Re}(f) \geq 0$ in some punctured neighborhood of p. Show that p is a removable singularity.

6. Suppose $\mathcal{F}$ is a normal family of holomorphic functions defined in a fixed domain. Show that for each integer $k \geq 1$ the k-th derivative family $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal.