Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
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Notations

- $\mathbb{C}$: the complex plane
- $\Delta = \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω
- $B(0, R) := \{z \in \mathbb{C} : |z| < R\}$
- $\overline{B(0, R)} := \{z \in \mathbb{C} : |z| \leq R\}$
- By a "conformal map" f of a region V_1 to a region V_2 we mean a one-to-one holomorphic map f of V_1 onto V_2.
- By a "region" G we mean a nonempty connected open set in $\mathbb{C}$.

PART I: Answer Any TWO Questions.

1. Let G be a simply connected region in $\mathbb{C}$. Suppose that $f \in \mathcal{O}(G)$ such that $f(z) \neq 0$ for any z in G. Show that there exists $g \in \mathcal{O}(G)$ such that $f = e^g$.

2. State and prove Schwarz's lemma.

3. a) Define an elliptic function.
b) Prove that an elliptic function cannot be holomorphic unless it is a constant.

4. Let u be a real valued harmonic function in $\{z : 0 < |z| < 1\}$. If u is bounded, prove that u can be extended to a harmonic function in Δ.

PART II: Answer Any TWO Questions.

1. Suppose $f \in \mathrm{Aut}(\Delta)$ such that f has two (distinct) fixed points in Δ. Show that f must be the identity.

2. Suppose $f_1 : \Delta \to \Omega_1$ and $f_2 : \Delta \to \Omega_2$ are two conformal maps, where Ω_i, $i = 1, 2$, are simply connected regions. Suppose $\Omega_1 \subset \Omega_2$ and $f_1(0) = f_2(0)$. Show that

$$|f_1'(0)| \leq |f_2'(0)|$$

and that equality holds if and only if $\Omega_1 = \Omega_2$

3. Let f and g be entire functions such that, for all $z \in \mathbb{C}$,

$$e^{f(z)} + e^{g(z)} = 1.$$

Prove that both f and g are constants.

PART III: Answer Any FOUR Questions.

1. Suppose f and g are continuous on $\overline{B(0,R)}$, and holomorphic in $B(0,R)$, with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither f nor g vanishes in $\overline{B(0,R)}$, there exists a constant λ, with $|\lambda| = 1$, such that $f = \lambda g$.

2. Show that if f and g are holomorphic functions in a region G, such that $\overline{f}g$ is holomorphic, then either f is a constant or $g(z) = 0$ for all z in G.

3. Find the number of zeroes of the equation

$$z^9 + z^4 - 8z^3 + 2z + 1 = 0$$

in the annulus $\{z : 1 < |z| < 2\}$. Show **all** your work.

4. Prove that there exists a sequence of polynomials $P_n(z)$ such that $P_n(0) = 1$ for $n \in \mathbb{N}$ and $P_n(z) \to 0$ for every $z \neq 0$ as $n \to \infty$.

5. Find a conformal mapping from the region Ω outside of the ellipse

$$\frac{x^2}{25/16} + \frac{y^2}{9/16} = 1$$

onto itself such that ∞ is preserved and $5/4$ is mapped onto $(3/4)i$.

6. Let f be an entire function with negative imaginary part. Prove that f must be a constant function.

7a. Find $g \in \mathcal{O}(\mathbb{H})$ whose set of zeroes is $\{\frac{i}{n} : n \in \mathbb{N}\}$, where $\mathbb{H} = \{z = x + iy : y > 0\}$.

OR

7b. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$