Differential Geometry
Spring 2016/F2016

Do any 6 problems. *Note:* Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ unless it is specifically stated otherwise.

1. Suppose that ω is a smooth differential form of degree one on a manifold M. Show that if ω does not vanish anywhere on M then there exists a smooth vector field X on M such that $\omega(X) \equiv 1$.

2. Let $G = SO(n, \mathbb{R})$, $n \geq 2$, be the group of $n \times n$ orthogonal matrices of determinant one.

   a) Show that G acts transitively on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

   b) Identify the isotropy group of the point $(1, 0, \dots, 0)^t$.

   c) Prove that G is connected for every $n \geq 2$.

3. Prove that the tangent bundle of every connected, smooth manifold is orientable.

4. Given $\varepsilon > 0$, consider the rank-two distribution

$$\mathcal{D}_\varepsilon = \operatorname{span}(\partial_x, \partial_y + \varepsilon x \partial_z) = \ker(dz - \varepsilon x dy) \text{ on } \mathbb{R}^3.$$

   (a) Check that $\mathcal{D}_\varepsilon$ is not integrable.

   (b) Given $z > 0$, show that there exist $T > 0$ and a curve $\gamma : [0, T] \to \mathbb{R}^3$ with $\gamma(0) = (0, 0, 0)$ and $\gamma(T) = (0, 0, z)$ such that $\dot\gamma(t) \in \mathcal{D}_\varepsilon$ for all $t \in [0, T]$.

   Hint: Use your answer in item (a).

   (c) Show that γ as in (b) can be chosen in such a way that its length $L(\gamma)$ with respect to the metric $(dx)^2 + (dy^2) + (dz - \varepsilon x dy)^2$ satisfies $L(\gamma) \leq C\sqrt{z/\varepsilon}$ for some uniform constant C.

   (d) Show that $L(\gamma) \geq c\sqrt{z/\varepsilon}$ for all curves as in (b), and some (other) uniform constant $c > 0$.

   Hint: $dz = \varepsilon x dy$ on γ, so $z = \varepsilon \int_{\bar\gamma} x dy$, where $\bar\gamma$ is the projection of γ to the (x, y)-plane.

5. For any integer $0 \leq k \leq n$, let $G_{k,n} = G_k(\mathbb{R}^n)$ denote the set of all k-dimensional linear subspaces of $\mathbb{R}^n$.

   Prove that $G_{k,n}$ can be naturally given the structure of a smooth manifold of dimension $k(n-k)$. Also, show that $G_{k,n}$ and $G_{n-k,n}$ are diffeomorphic.

6. Let M be a compact manifold of dimension $2n$. Let ω be a 2-form on M such that the induced bundle map $\tilde{\omega} : TM \to T^*M$ defined by $\tilde{\omega}(X)(Y) = \omega(X, Y)$ is a bundle isomorphism. Show that if $d\omega = 0$, then $[\omega^n] \in H^{2n}_{dR}(M)$ is non-zero.

7. Let G be a Lie group and let H be Lie a subgroup of dimension k. Let $L_g : G \to G$ be given by $L_g(h) = gh$. Show there is a unique rank k distribution D on G with the property that

$$(L_g)_*(TH) \subset D \quad \text{for all } g \in G.$$

Show that the distribution D is integrable and identify its integral submanifolds.

8. Let M, N, and P be smooth manifolds. Let $f : M \to P$ and $g : N \to P$ be surjective submersions. Let

$$M \times_P N = \{(m, n) \in M \times N |\, f(m) = g(n)\},$$

and let $\pi_1 : M \times_P N \to M$ be $\pi_1(m, n) = m$ and $\pi_2 : M \times_P N \to N$ be $\pi_2(m, n) = n$. Show that $M \times_P N$ is a smooth manifold, that $\pi_1 \circ f = \pi_2 \circ g$, and that this is a surjective submersion.

9. Let T be a relatively compact domain, diffeomorphic to the 2-disk, in a Riemannian surface M, with boundary of T consisting of five geodesic segments. Assume that the Gauss curvature is non-positive at every point of T. Give an upper bound on the sum of interior angles of the geodesic 5-sided polygon T.

10. Let M be a complete Riemannian manifold and $\gamma : (-1, 1) \to M$ a smooth curve. Suppose X_s is a vector field along γ such that $|X_s| \equiv 1$ and $\langle X_s, \dot{\gamma}(s)\rangle \equiv 0$. Consider the mapping $E : (-1, 1) \times (-\infty, \infty) \to M$ given by

$$E(s, t) = \exp_{\gamma(s)}(tX_s).$$

Show that for every (s_0, t_0) the curve $s \to E(s, t_0)$ is perpendicular to the geodesic $t \to E(s_0, t)$ at (s_0, t_0).