Differential Geometry
Fall 2019

Do any 6 problems. *Note:* Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ unless it is specifically stated otherwise.

1. Consider $\mathbb{R}^2$ endowed with the metric $dx \otimes dx + e^{2x} dy \otimes dy$.

   (a) Show that this metric has sectional curvature -1.

   (b) Show that for each $A, B \in \mathbb{R}$, the curve $\gamma(s) = (s + A, B)$ is a geodesic.

2. Let M be a Riemannian manifold and $\gamma : [0, a] \to M$ a smooth curve parametrized by arc length. Let V be a smooth vector field along γ vanishing at the endpoints, i.e. $V(0) = V(a) = 0$.

   (a) Prove that for some $\epsilon > 0$ there exists a map $\alpha : [0, a] \times (-\epsilon, \epsilon) \to M$ which is a proper variation of γ, i.e. $\alpha(t, 0) = \gamma(t)$ for all $t \in [0, a]$, and $\alpha(0, s) = \gamma(0)$, $\alpha(a, s) = \gamma(a)$ for all $s \in (-\epsilon, \epsilon)$, and such that $\frac{\partial \alpha}{\partial s}(t, 0) = V(t)$ for all $t \in [0, a]$.

   (b) If $\gamma_s(\cdot) := \alpha(\cdot, s)$, prove the first variation of arc length formula:

$$\frac{d}{ds}_{|s=0} L(\gamma_s) = -\int_0^a \left\langle V(t), \frac{D\gamma'(t)}{dt} \right\rangle \, dt.$$

3. (a) State the inverse function theorem.

   (b) Prove that $f(r, \theta) = (r \cos(\theta), r \sin(\theta))$ is a diffeomorphism from $(2, 3) \times (0, \pi)$ to $\{(x, y) : x^2 + y^2 \in (4, 9), y > 0\}$, but it is not a diffeomorphism from $(2, 3) \times (0, 3\pi)$ to $\{(x, y) : x^2 + y^2 \in (4, 9)\}$.

4. Consider the half-plane $G := \{(x, y) \in \mathbb{R}^2 \, : \, y > 0\}$ endowed with the binary operation:

$$(a, b) \star (a', b') = (a + ba', bb').$$

   (a) Show that G is a Lie group.

   (b) Define what is an orientable manifold.

   (c) Prove that G is orientable and more generally prove that a Lie group is always orientable.

5. (a) Consider the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (xy, 1).$$

   Compute: $f^*(dx),\ f^*(dy),\ f^*(xdy),\ f^*(ydx)$.

   (b) Consider the differential form $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ in $\mathbb{R}^3$. Prove that ω is invariant under rotations, i.e. if A is an orthogonal transformation of $\mathbb{R}^3$ then $A^*\omega = \omega$, and that

$$\int_{S^2} \omega > 0.$$

6. Let S be a surface equipped with a complete Riemannian metric and $\gamma : (-1,1) \to S$ a smooth curve. Suppose X_t is a parallel vector field along γ such that $|X_t| \equiv 1$ and $\langle X_t, \dot{\gamma}(t)\rangle \equiv 0$. Consider the mapping $E : (-1,1) \times (-\infty, \infty) \to S$ given by

$$E(s,t) = \exp_{\gamma(t)}(sX_t).$$

Show that the curves $t \to E(s_0, t)$ are perpendicular to geodesics $s \to E(s, t_0)$ for all (s_0, t_0).

7. Consider the mapping $F : \mathbb{R}^3 \to \mathbb{R}^6$ given by the formula

$$F(x,y,z) = (x^2, y^2, z^2, yz, xz, xy).$$

   (a) Show that F is an immersion everywhere except at a point in $\mathbb{R}^3$.
   (b) Show that F restricted to $S^2 \subset \mathbb{R}^3$ is an immersion.
   (c) Show that F induces a map $\tilde{F}$ from the real projective plane $\mathbb{R}P^2$ to $\mathbb{R}^6$.
   (d) Show that the image of $\tilde{F}$ is contained in the hyperplane

$$\{(w_1, w_2, \ldots, w_6) : w_1 + w_2 + w_3 = 1\} \subset \mathbb{R}^6.$$

   (e) Show that F is injective.

8. If M is a Riemannian manifold, the *gradient*, denoted $\operatorname{grad} : C^\infty(M) \to \mathfrak{X}(M)$ is defined by

$$\operatorname{grad} f = (df)^\sharp \quad (\text{i.e. } \langle \operatorname{grad} f, Y\rangle = df(Y) \text{ for all } Y \in \mathfrak{X}(M)).$$

   (a) Prove that in local coordinates,

$$\operatorname{grad} f = \sum_{i,j=1}^{n} (g^{-1})^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

   where g is the matrix whose ij-entry is $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$, and g^{-1} is its inverse.
   (b) Prove that if $E_1, \ldots, E_n$ is a local orthonormal frame, then $\operatorname{grad} f = \sum_{i=1}^n E_i(f)\, E_i$, which is the usual formula for the gradient in $\mathbb{R}^n$.

9. Let $M = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|\boldsymbol{x} - \boldsymbol{y}\| = 1\}$.

   (a) Show that M is a submanifold of $\mathbb{R}^6$ of dimension 5.
   (b) Show that M is diffeomorphic to $\mathbb{R}^3 \times S^2$.

10. Let G be a connected Lie group with a bi-invariant metric g. Let ∇ denote the Levi-Civita connection. Then it is known that for left invariant vector fields X, Y, Z one has

$$g([X,Y],Z) = g(X,[Y,Z]) \quad \text{and} \quad \nabla_X Y = \frac{1}{2}[X,Y].$$

   (a) Show that the curvature tensor of ∇ satisfies

$$R(X,Y,Z,W) = -\frac{1}{4}\, g([X,Y],[Z,W]),$$

   where X, Y, Z, W are left-invariant vector fields.
   (b) Show that the sectional curvature of G is non-negative.
   (c) Prove that the Lie algebra $\mathfrak{g}$ of G has zero bracket if and only if the sectional curvature of (G, g) is zero.