Instructions: No more than 6 problems will be graded—specify which ones you want graded.

Note: Throughout this exam, all manifolds are $C^\infty$ and connected, and all maps are $C^\infty$ unless it is specified otherwise.

1. Let $p : \mathbb{R}^{n+k} \to \mathbb{R}^n, (x_1, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_n)$ be the projection map. Prove that the forms on $\mathbb{R}^{n+k}$ that are pullbacks of forms on $\mathbb{R}^n$ are exactly those that are in the kernel of the interior multiplications by $\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{n+k}}$ and whose $d$ is also in the kernel of the interior multiplications by $\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{n+k}}$.

2. Let $\alpha$ be a 1-form in $\mathbb{R}^3$. Show that if $\alpha$ is invariant under all isometries of $\mathbb{R}^3$, then $\alpha$ must be zero.

3. For $n \geq 1$ denote by $S^n \subset \mathbb{R}^{n+1}$ the $n$-sphere, and define $f : S^n \to S^n, f(x) := -x$ to be the antipodal map. Show that $f$ is orientation-preserving iff $n$ is odd.

4. Let $(M, g = \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. For a function $f : M \to \mathbb{R}$, the gradient of $f$ is the vector field $\text{grad}(f) \in \mathfrak{X}(M)$ defined by
$$\langle \text{grad}(f), X \rangle = df(X) = X(f) \quad \forall X \in \mathfrak{X}(M)$$
For a vector field $V \in \mathfrak{X}(M)$, the curl$(V)$ is the $(0, 2)$-tensor field defined by
$$\text{curl}(V)(X, Y) = \langle \nabla_X V, Y \rangle - \langle \nabla_Y V, X \rangle \quad \forall X, Y \in \mathfrak{X}(M)$$
Show that $\text{curl}(\text{grad}(f)) = 0$.

5. Let $M$ be the surface $M := \{(s, t, s \cdot t) \mid s, t \in \mathbb{R}\} \subset \mathbb{R}^3$.
   (a) Show that at $p = (s, t, s \cdot t) \in M$, the vectors $(1, 0, t) \in T_p\mathbb{R}^3$ and $(0, 1, s) \in T_p\mathbb{R}^3$ are tangent vectors of $M$, and, furthermore, that $N = \frac{(t, s, -1)}{\sqrt{1 + s^2 + t^2}} \in T_p\mathbb{R}^3$ is a unit normal vector to $M$.
   (b) Show that the Gaussian curvature of $M$ is $K = \frac{-1}{(1 + s^2 + t^2)^2}$.

6. Consider the metric on $\mathbb{R}^2$ given by
$$dx \otimes dx + (1 + x^2)^2 \cdot dy \otimes dy$$
Find the Gaussian curvature of this metric.
(7) Denote by $G := \{A \in GL(n, \mathbb{R}) \mid \exists c > 0 : A^{-1} = c \cdot A^t\}$.

(a) Show that $G$ with the matrix multiplication is a Lie group by showing that it is isomorphic as a Lie group to $(O(n), \cdot) \times (\mathbb{R}, +)$.
(b) Identify the Lie algebra $\mathfrak{g}$ of $G$ as a sub-Lie algebra of $\mathfrak{gl}(n, \mathbb{R})$.

(8) Let $\{U_i\}_{i \in I}$ be an open cover of a manifold $M$, let $V$ be a vector space, and let $\{g_{ij} : U_i \cap U_j \to Aut(V)\}_{ij}$ be the transition functions for a vector bundle $E$ with fiber $V$. Furthermore, let $\{A_i \in \Omega^1(U_i, End(V))\}_i$ be the 1-forms of a connection on $E$, i.e., the $A_i$ satisfy

$$A_i = g_{ji}^{-1} \cdot A_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} \quad \text{on } U_i \cap U_j.$$ 

If $\{f_i : U_i \to Aut(V)\}_i$ are such that for all $i, j$, the identity $g_{ji} \cdot f_i = f_j \cdot g_{ji}$ holds on $U_i \cap U_j$, then show that

$$B_i := f_i^{-1} \cdot A_i \cdot f_i + f_i^{-1} \cdot df_i$$

are also the 1-forms of a connection on $E$, i.e., they satisfy

$$B_i = g_{ji}^{-1} \cdot B_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} \quad \text{on } U_i \cap U_j.$$ 

(9) Let $E \to M$ be a smooth vector bundle with affine connection $\nabla$. Denote by $\Omega$ its curvature 2-form. Show that the trace of the curvature, $tr(\Omega)$, is a closed, globally defined 2-form.

Hint 1: You may use without proof that $d$ commutes with the trace.

Hint 2: You may use without proof that for a matrix of $p$-forms $A$ and a matrix of $q$-forms $B$, we have $tr(A \wedge B) = (-1)^{p+q} \cdot tr(B \wedge A)$.

(10) Let $(M, g)$ be a Riemannian manifold. For a given chart of $M$ denote by $g_{ij}$ the components of the metric tensor $g$, and denote by $\Gamma^k_{ij}$ the Christoffel symbols of the Levi-Civita connection. Prove the following identities:

(a) $\Gamma^k_{ij} = \Gamma^k_{ji}$
(b) $\frac{\partial g_{ij}}{\partial x^k} = g_{kl} \Gamma^l_{ki} + g_{kl} \Gamma^l_{kj}$
(c) Use (a) and (b) to show: $\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^k} \right)$

(11) Let $T$ be a regular tetrahedron.

(a) Calculate the Euler characteristic of $T$.
(b) Confirm the combinatorial Gauss-Bonnet theorem for $T$. 