DIFFERENTIAL GEOMETRY QUALIFYING EXAM
SPRING 2021

Instructions: No more than 6 problems will be graded—specify which ones you want graded.
Note: Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ unless it is specified otherwise.

(1) Consider $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4. Prove that on $\mathbb{R}^4$, the 2-form $dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ is not a wedge product $\alpha \wedge \beta$ of two 1-forms $\alpha, \beta \in \Omega^1(\mathbb{R}^4)$.

(2) Let $\pi : X \to Y$ be a surjective submersion.
  (a) Show that a k-form $\alpha \in \Omega^k(Y)$ is closed $\Leftrightarrow$ $\pi^*(\alpha)$ is closed.
  (b) Give a counterexample for the statement that α is exact $\Leftrightarrow$ $\pi^*(\alpha)$ is exact.

(3) Denote by $T^2 \subseteq \mathbb{R}^3$ the surface of revolution obtained by rotating $x^2 + (y-2)^2 = 1$ about the x-axis. Pick an orientation given by a unit normal vector field along T^2. Show that the orientation considered as a map $T^2 \to S^2$ is onto.

(4) Find the circumference of the circle of radius 1 around the origin in the metric on $\mathbb{R}^2$ given by

$$\frac{4}{(1+x^2+y^2)^2} \cdot \Big(dx \otimes dx + dy \otimes dy \Big)$$

(5) For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ denote by $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$ as usual. Consider a metric on $\mathbb{R}^n$ of the form

$$f(|x|) \cdot (dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)$$

for some function $f : [0,\infty) \to (0,\infty)$, which is non-decreasing on $[0,1]$. Prove that for any piecewise smooth curve $\gamma : [a,b] \to \mathbb{R}^n$ with $\gamma(a) = (0,0,\ldots,0)$ and $\gamma(b) = (1,0,\ldots,0)$, the length of γ is greater than or equal to the length of the straight line path $\lambda(t) = (t,0,\ldots,0)$ for $t \in [0,1]$.

(6) Let $h : \mathbb{R} \to (0,\infty)$ be a smooth function. Consider the metric on $\mathbb{R}^2$ given by

$$dx \otimes dx + h(x)^2 \cdot dy \otimes dy.$$

Show that the Gaussian curvature of this metric is:

$$K(x,y) = \frac{-h''(x)}{h(x)}$$

(7) Let $\{U_i\}_{i\in I}$ be an open cover of a manifold M, and let $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R})$ be the transition functions for a vector bundle. For $i \in I$, let $\chi_i : M \to \mathbb{R}$ be a smooth function so that each χ_i has support in U_i, the $\{\chi_i\}_{i\in I}$ are locally finite, and $\sum_{i\in I} \chi_i = 1$. Define A_i on U_i to be given by

$$A_i = \sum_j \chi_j g_{ji}^{-1} dg_{ji}.$$

Show that this defines a connection on the vector bundle, which means that the $\{A_i\}_{i\in I}$ on $U_i \cap U_j$ satisfy:

$$A_i = g_{ji}^{-1} A_j g_{ji} + g_{ji}^{-1} dg_{ji}$$

(8) Let $\{U_i\}_{i\in I}$ be an open cover of a manifold M, and let $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R})$ be the transition functions for a vector bundle. Show that:

(a) On $U_i \cap U_j \cap U_k$ we have:

$$tr(g_{ji}^{-1} dg_{ji}) + tr(g_{kj}^{-1} dg_{kj}) - tr(g_{ki}^{-1} dg_{ki}) = 0$$

Hint 1: Consider $g_{kj} g_{ji} = g_{ki}$.
Hint 2: You may use without proof that for a matrix of p-forms A and a matrix of q-forms B, we have $tr(A \wedge B) = (-1)^{p\cdot q} \cdot tr(B \wedge A)$.

(b) On $U_i \cap U_j$ we have:

$$d(g_{ij}^{-1}) = -g_{ij}^{-1} \cdot d(g_{ij}) \cdot g_{ij}^{-1}$$

(c) Conclude from (b), that on $U_i \cap U_j$:

$$d(tr(g_{ij}^{-1} dg_{ij})) = 0$$

Hint: You may use without proof that d commutes with the trace.

(9) Let G be a Lie group. For $g \in G$, denote by $\psi : G \to G, \psi(g) = g^{-1}$ the inverse map, and denote by $L_g : G \to G, L_g(h) = g \cdot h$ and $R_g : G \to G, R_g(h) = h \cdot g$ the left- and right-multiplication by g, respectively.

(a) Show that for $g \in G$: $\quad R_{g^{-1}} \circ \psi = \psi \circ L_g$

(b) Suppose there is a metric σ on G so that ψ is an isometry. Show that:

$$L_g^*(\sigma) = \sigma \text{ for all } g \in G \quad \text{iff} \quad R_g^*(\sigma) = \sigma \text{ for all } g \in G$$

(10) Recall that the 2-sphere $S_r^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2\}$ of radius r has a constant Gaussian curvature of $1/r^2$. Using the parametrization of S_r^2 given by

$$F(\varphi, \theta) = (r \sin(\theta) \sin(\varphi), r \sin(\theta) \cos(\varphi), r \cos(\theta)),$$

for $0 < \varphi < 2\pi$ and $0 < \theta < \pi$, fix an angle $0 < \alpha < 2\pi$ and denote by Ω_α the image of F restricted to $0 < \varphi < \alpha$ and $0 < \theta < \frac{\pi}{2}$,

$$\Omega_\alpha = F\left((0, \alpha) \times (0, \frac{\pi}{2})\right).$$

Verify the Gauss-Bonnet formula for the region Ω_α by checking that both sides of the equation coincide.
Hint: You may use the fact that the area of S_r^2 is $4\pi r^2$.