(1) Consider $\mathbb{R}^4$ with coordinates $x^1, x^2, x^3, x^4$. Prove that on $\mathbb{R}^4$, the 2-form $dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ is not a wedge product $\alpha \wedge \beta$ of two 1-forms $\alpha, \beta \in \Omega^1(\mathbb{R}^4)$.

(2) Let $\pi : X \to Y$ be a surjective submersion.
   (a) Show that a $k$-form $\alpha \in \Omega^k(Y)$ is closed $\iff \pi^*(\alpha)$ is closed.
   (b) Give a counterexample for the statement that $\alpha$ is exact $\iff \pi^*(\alpha)$ is exact.

(3) Denote by $T^2 \subseteq \mathbb{R}^3$ the surface of revolution obtained by rotating $x^2 + (y - 2)^2 = 1$ about the $x$-axis. Pick an orientation given by a unit normal vector field along $T^2$. Show that the orientation considered as a map $T^2 \to S^2$ is onto.

(4) Find the circumference of the circle of radius 1 around the origin in the metric on $\mathbb{R}^2$ given by
   $$\frac{4}{(1 + x^2 + y^2)^2} \cdot (dx \otimes dx + dy \otimes dy)$$

(5) For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ denote by $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$ as usual. Consider a metric on $\mathbb{R}^n$ of the form
   $$f(|x|) \cdot (dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)$$
   for some function $f : [0, \infty) \to (0, \infty)$, which is non-decreasing on $[0, 1]$. Prove that for any piecewise smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = (0, 0, \ldots, 0)$ and $\gamma(b) = (1, 0, \ldots, 0)$, the length of $\gamma$ is greater than or equal to the length of the straight line path $\lambda(t) = (t, 0, \ldots, 0)$ for $t \in [0, 1]$.

(6) Let $h : \mathbb{R} \to (0, \infty)$ be a smooth function. Consider the metric on $\mathbb{R}^2$ given by
   $$dx \otimes dx + h(x)^2 \cdot dy \otimes dy.$$
   Show that the Gaussian curvature of this metric is:
   $$K(x, y) = \frac{-h''(x)}{h(x)}.$$
(7) Let \( \{U_i\}_{i \in I} \) be an open cover of a manifold \( M \), and let \( g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R}) \) be the transition functions for a vector bundle. For \( i \in I \), let \( \chi_i : M \to \mathbb{R} \) be a smooth function so that each \( \chi_i \) has support in \( U_i \), the \( \{\chi_i\}_{i \in I} \) are locally finite, and \( \sum_{i \in I} \chi_i = 1 \). Define \( A_i \) on \( U_i \) to be given by

\[
A_i = \sum_j \chi_j g_{ji}^{-1} dg_{ji}.
\]

Show that this defines a connection on the vector bundle, which means that the \( \{A_i\}_{i \in I} \) on \( U_i \cap U_j \) satisfy:

\[
A_i = g_{ji}^{-1} A_j g_{ji} + g_{ji}^{-1} dg_{ji}
\]

(8) Let \( \{U_i\}_{i \in I} \) be an open cover of a manifold \( M \), and let \( g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R}) \) be the transition functions for a vector bundle. Show that:

(a) On \( U_i \cap U_j \cap U_k \) we have:

\[
tr(g_{ji}^{-1} dg_{ji}) + tr(g_{kj}^{-1} dg_{kj}) - tr(g_{ki}^{-1} dg_{ki}) = 0
\]

Hint 1: Consider \( g_{kj}^{-1} g_{ji} = g_{ki} \).

Hint 2: You may use without proof that for a matrix of \( p \)-forms \( A \) and a matrix of \( q \)-forms \( B \), we have \( tr(A \wedge B) = (-1)^{p-q} \cdot tr(B \wedge A) \).

(b) On \( U_i \cap U_j \) we have:

\[
d(g_{ij}^{-1}) = -g_{ij}^{-1} \cdot d(g_{ij}) \cdot g_{ij}^{-1}
\]

(c) Conclude from (b), that on \( U_i \cap U_j \):

\[
d(tr(g_{ij}^{-1} dg_{ij})) = 0
\]

Hint: You may use without proof that \( d \) commutes with the trace.

(9) Let \( G \) be a Lie group. For \( g \in G \), denote by \( \psi : G \to G, \psi(g) = g^{-1} \) the inverse map, and denote by \( L_g : G \to G, L_g(h) = g \cdot h \) and \( R_g : G \to G, R_g(h) = h \cdot g \) the left- and right-multiplication by \( g \), respectively.

(a) Show that for \( g \in G \):

\[
R_{g^{-1}} \circ \psi = \psi \circ L_g
\]

(b) Suppose there is a metric \( \sigma \) on \( G \) so that \( \psi \) is an isometry. Show that:

\[
L_g^*(\sigma) = \sigma \text{ for all } g \in G \quad \text{iff} \quad R_g^*(\sigma) = \sigma \text{ for all } g \in G
\]

(10) Recall that the 2-sphere \( S_r^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2 \} \) of radius \( r \) has a constant Gaussian curvature of \( 1/r^2 \). Using the parametrization of \( S_r^2 \) given by

\[
F(\varphi, \theta) = (r \sin(\theta) \sin(\varphi), r \sin(\theta) \cos(\varphi), r \cos(\theta))
\]

for \( 0 < \varphi < 2\pi \) and \( 0 < \theta < \pi \), fix an angle \( 0 < \alpha < 2\pi \) and denote by \( \Omega_\alpha \) the image of \( F \) restricted to \( 0 < \varphi < \alpha \) and \( 0 < \theta < \frac{\pi}{2} \),

\[
\Omega_\alpha = F \left( (0, \alpha) \times (0, \frac{\pi}{2}) \right).
\]

Verify the Gauss-Bonnet formula for the region \( \Omega_\alpha \) by checking that both sides of the equation coincide.

Hint: You may use the fact that the area of \( S_r^2 \) is \( 4\pi r^2 \).