**MATH 81600. Probability I.**

**Schedule:** M 4:15 - 6:15 p.m.; Room 5417

**Instructor:** Elena Kosygina
office: 6-245, (646) 312-4167 (Baruch); 4214.07, (212) 817-8575 (GC);
email: elena.kosygina@baruch.cuny.edu

**Prerequisites:** Functions of a Real Variable (70100 and 70200 or equivalent).

**Course description:** The course will cover basics probability measures, random variables, expectations, weak and strong laws of large numbers, weak convergence, characteriatic functions, central limit theorems, conditional expectations, and martingales. Time permitting, other topics (e.g. Markov chains, ergodic theorems) might be included.

Below is a list of review topics for the final exam in one of the previous years (when the course was 4.5 credits; references are to Durrett's 2nd edition, typically you have to add 1 to the first number to convert to the 4th edition reference). It was not intended to be comprehensive but was made rather as an aid for the exam preparation. The course every year is somewhat different.

**References:**

- R. Durrett, Probability: Theory and Examples, Version 4.1, https://www.math.duke.edu/∼rtd/PTE/pte.html
- A. Dembo, Probability Theory, statweb.stanford.edu/∼adembo/nyu-2911/lnotes.pdf
- S.R.S. Varadhan, Probability Theory, Courant Lecture Notes 7, AMS, 2001
- O. Kallenberg, Foundations of Modern Probability, 2nd edition, Springer 2001.

**Workload and grading:** Weekly homework (selected problems will be collected and graded), the midterm exam, and the final exam. Count on having in class exams. Grade will be based equally on the above 3 components.

**Final exam review sheet from one of the previous years**

## 1. Definitions (with basic examples/counterexamples) of

- $\sigma$-algebra, $\pi$-system, $\lambda$-system;
- measurable space, probability measure, probability space, finite/countable product of probability spaces, consistent set of probability measures (see Kolmogorov's extension theorem);

- random variable/vector, $\sigma$-algebra generated by a random variable, distribution measure, distribution function of a random variable, identically distributed random variables, density, joint density, joint distribution measure, joint distribution function function, absolutely continuous and singular distributions, marginal distribution;
- uniform integrability;
- almost sure convergence, convergence in $L^p$, convergence in probability, convergence in distribution, weak convergence;
- lower and upper semicontinuous functions, convex functions;
- expectation, variance, standard deviation, $n$-th moment of a random variable, covariance and correlation of random variables
- independence of any number of events, random variables, $\sigma$-algebras;
- tightness of a family of probability measures, relative compactness of a family of probability measures;
- characteristic function of a random variable/probability distribution and its basic properties. Examples of characteristic functions (for Bernoulli, Poisson, normal, uniform on $[a, b]$, exponential, normal, bilateral, and Cauchy distributions);
- Poisson process with rate $\lambda$, Poisson process on a general measure space;
- stopping times, associated $\sigma$-algebras and their properties (operations on stopping times, inclusions for $\sigma$-algebras);
- tail $\sigma$-algebra, exchangeable $\sigma$-algebra;
- recurrent and possible values for a random walk, recurrent and transient random walks;
- conditional expectation of a r.v. and its basic properties (with proofs);
- regular conditional distributions and regular conditional probabilities;
- martingale, super- and sub- martingale;

## 2. Distributions (with examples and interpretations of parameters)

- Discrete distributions: Bernoulli, binomial, geometric, Poisson, uniform on a finite set, negative binomial*, hypergeometric*
- Continuous distributions: uniform on $[a, b]$, exponential, normal, Cauchy, gamma, lognormal, Rayleigh*, ($\beta$, $\chi^2$, $F$, $T$ - used in statistics, all optional)

*-red distributions are listed just for your information. There are, of course, many more famous distributions.

**3. Theorems and facts (with applications)**

- Conditions equivalent to the countable additivity (monotonicity from above, from below, and "at zero", exer. 1.1.1). Countable subadditivity and Bonferroni inequalities.
- For every distribution function there is a r.v. which has that distribution.
- Every $\sigma(X)$-measurable random variable $Y$ is a Borel function of $X$ (exer. 1.2.8).
- properties of expectation, Jensen, Hölder, Markov, Chebyshev inequalities, change of variables formula.
- Fatou's, monotone, bounded, dominated convergence theorems (with the appropriate convergence in assumptions), Theorem (3.8), exer. 2.2.5.
- Dynkin's $\pi - \lambda$ theorem or any version of a monotone class theorem (without a proof), Corollary 4.5 (functions of disjoint sets of independent random variables are independent), joint distributions, densities of independent random variables (i.r.v.), computing expectations of functions of i.r.v., distributions of sums of i.r.v..
- Kolmogorov's extension theorem (without a proof) and exer. 1.4.20.
- If $Y \geq 0$ and $p > 0$ then $EY^p = \int_0^\infty py^{p-1}P(Y > y)\,dy$ and its generalization and discrete analogs (Lemma 1.5.7, exer. 1.5.6, 1.5.7).
- Various Weak Laws of Large Numbers (mainly $L^2$ and the necessary and sufficient condition of the WLLN for i.i.d. r.v.)
- Borel-Cantelli Lemmas.
- Strong Law of Large Numbers.
- Renewal Theorem (Theorem 1.7.3).
- Glivenko-Cantelli Theorem (without a proof).
- Equivalent definition of weak convergence/convergence in distribution. Continuous mapping theorem.
- Helly's selection theorem.
- Prokhorov's Theorem (proof for $\mathbb{R}^n$).
- The inversion formula for a characteristic function (Theorems 2.3.2 and 2.3.3, any one of them with a proof).
- Continuity theorem (without proof).

- Connection between the smoothness of the characteristic function of distribution $\mu$ and the decay of $\mu$ at infinity (Section 2.3c).
- The De Moivre-Laplace Theorem (the normal approximation to binomial distribution).
- The Central Limit Theorem for i.i.d. sequences.
- The Lindeberg-Feller Theorem (without proof) and applications (exer. 2.4.10 and Lyapunov's Theorem)
- Berry-Esseen Theorem (without proof).
- Poisson convergence theorem (Poisson approximation to an array of binomial distributions). Application: construction of a Poisson process with rate $\lambda$.
- Kolmogorov and Hewitt-Savage $0-1$ laws. Application to random walks on $\mathbb{R}$.
- The first and the second Wald's equations.
- Equivalent conditions for recurrence of random walks on $\mathbb{Z}^d$ (Theorem 3.2.2.). Recurrence and transience of simple random walks on $\mathbb{Z}^d$.
- Reflection principle for a simple random walk on $\mathbb{Z}$. Applications: the Ballot Theorem, the distribution of time to hit 0, Arcsine law (the last 2 without proof).
- Conditional expectation of a square integrable r.v. and an orthogonal projection (Theorem 4.1.4.).
- Optional stopping theorems.
- Martingale convergence theorem, Doob's inequality, convergence in $L^p$.