Logic Qualifying Exam
Three Parts
Autumn 2017

Part Zero

Answer all of the following questions.

1. Let $(A, <)$ and $(B, <)$ be well-orders. Assume that each of these structures is isomorphic to some initial segment of the other. Prove, directly from the definition of well-order, that they are isomorphic to each other.

2. State the Compactness Theorem (also known as the Finiteness Theorem) correctly, and use it to sketch a proof that it is consistent with the theory of the structure $(\mathbb{N}, 0, 1, <, +, \cdot)$ for a number to have infinitely many prime factors.

3. (a) Let L be a first order language without constant or function symbols. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures and that $f : M \to N$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. Show that if $\varphi(x_1 \ldots x_n)$ is an L-formula and $a_1 \ldots a_n \in M$ then $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \varphi(f(a_1), \ldots, f(a_n))$.

   (b) If, in part (a), the map f is only a homomorphism from $\mathcal{M}$ into $\mathcal{N}$, then for which formulas $\varphi(x_1 \ldots x_n)$ could the conclusion of part (a) fail? (By definition, a homomorphism has all the same properties as an isomorphism, except that it need not be surjective: it must be injective and must preserve all functions, relations, constants, and negations of relations.) Explain your answer briefly.

4. State Gödel's First and Second Incompleteness Theorems correctly, and sketch the proof of one of them. (A paragraph or so is sufficient, if written accurately.)

Part One

Do four of the following six problems.

1. Consider the following statements; prove the one(s) that are true, and give explicit examples disproving the one(s) that are not true.

   - Let $L := \{<\}$ be the language of orders. Up to L-isomorphism, there are exactly two L-structures $\mathcal{A}$ with $|A| = 2$.
   - If $\mathcal{A}$ is an L-structure, and Σ is a set of L-formulas, and each finite subset of Σ is satisfied by some element of A, then the whole Σ is satisfied by some element of A.

2. The *spectrum* of a theory T is the set of sizes of its finite models:

   $$\mathrm{Spec}(T) := \{n \in \mathbb{N} : \exists \mathcal{M} \models T \text{ s.t. } |M| = n\}.$$

   Find 5 theories T with 5 different spectra.

3. Prove the following statement, or give an explicit counterexample.
   For any L-structure $\mathcal{A}$ and any two elementary extensions $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$, there exists and L-structure $\mathcal{D}$ and elementary embeddings $\beta : \mathcal{B} \to \mathcal{D}$ and $\gamma : \mathcal{C} \to \mathcal{D}$ such that $\beta(a) = \gamma(a)$ for all $a \in A$.

4. Prove the following statement, or give an explicit counterexample.
   $(\mathbb{Q}, <)$ is elementarily equivalent to $(\mathbb{R}, <)$.

5. Use one of the following to prove the other. (Both are actually true, but that's harder to prove.)

   - Fix two first-order languages L_1 and L_2; and an L_i-theory T_i for each $i = 1, 2$. Suppose that there is no $L_1 \cup L_2$-structure satisfying $T_1 \cup T_2$. Then there is some $L_1 \cap L_2$-sentence θ such that $T_1 \models \theta$ and $T_2 \models \neg\theta$.
   - Let $L \subset L'$ be first-order languages; let T be an L'-theory; let $\varphi(x)$ be an L'-formula. Suppose that for any two models $\mathcal{A}_1$ and $\mathcal{A}_2$ of T with the same universe A, if the reducts of the two $\mathcal{A}_i$ to L are same structure, then for any $a \in A$, $\mathcal{A}_1 \models \varphi(a)$ if and only if $\mathcal{A}_2 \models \varphi(a)$.
     Then there is an L-formula $\psi(x)$ such that $T \models \forall x\, \varphi(x) \leftrightarrow \psi(x)$.

6. Let S be the signature of rings, and let $\mathcal{A}_n$ be the S-structure with universe $\mathbb{Z}/n\mathbb{Z}$ and the usual ring structure. Let $T := Th(\{\mathcal{A}_p : p \text{ is prime }\})$.

   (a) Show that T has an infinite model.

   (b) Is there a model of T that has a substructure isomorphic to $\mathbb{Q}$?

   (c) Is there a model of T that is elementarily equivalent to $\mathbb{Z}$?

Part Two

Do four of the following eight problems.

1. Define the halting problem and prove that it is computably enumerable, but not computably decidable.

2. Prove that if a theory is decidable, then it has a decidable completion.

3. Assume PA is consistent. Is there a consistent theory T extending PA that proves its own inconsistency? (That is, for which $T \vdash \neg\mathrm{Con}(T)$?)

4. Prove true or prove false: there is a universal total computable function, that is, a total computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that every total computable function occurs as f_n for some n, where f_n is the function defined by $f_n(m) = f(n, m)$.

5. Prove that the collection of Turing machine programs p which halt on exactly one input is not decidable.

6. Let TA be the set of Gödel codes of sentences true in the standard model of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. Show that if $A \subseteq \mathbb{N}$ is definable in $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, then $A <_T TA$. (Notation, $X <_T Y$ means that X is Turing computable from an oracle for Y, but not conversely.)

7. Consider the structure $\langle \mathbb{Z}, <, U \rangle$, with the usual $<$ relation and with a unary predicate U on the integers $\mathbb{Z}$. Show that if U is not periodic, then no two distinct points in the structure have the same 1-types.

8. Prove that there is a computable graph that contains a copy of every countable graph as an induced subgraph.