LOGIC QUALIFYING EXAM
AUGUST 2019

Part 0

For any **FOUR** of the following results, give the statement and provide a proof.

(1) Elemenary Chain lemma.

(2) Upward Löwenheim-Skolem theorem.

(3) Łoś's lemma for ultraproducts.

(4) König's lemma for trees.

(5) Cantor's theorem about the cardinality of power sets.

(6) The Schroeder-Bernstein Theorem about the existence of bijections.

Part 1

Do any **FOUR** of the following problems.

(1) Prove that every set of real numbers that is parametrically definable in $(\mathbb{R}, <)$ is the union of finitely many open intervals and points.

(2) Prove that the only sets of real numbers that are definable without parameters in $(\mathbb{R},+)$ are $\varnothing$, $\{0\}$, $\mathbb{R}$, and $\mathbb{R} \setminus \{0\}$.

(3) Let c be an element of a set I, and let $U = \{X \subseteq I : c \in X\}$. Prove that for any structure $\mathfrak{M}$, the ultrapower $\mathfrak{M}/U$ is isomorphic to $\mathfrak{M}$.

(4) Suppose that a theory T is axiomatized by a set of sentences $\{\varphi_n : n \in \omega\}$, and that for any n there is an m such that φ_m is not a consequence of $\{\varphi_i; i \leq n\}$. Prove that T is not finitely axiomatizable.

(5) Show that there is no first order sentence ϕ such that for every ordered set $(A, <)$, $(A, <) \models \phi$ if and only if $<$ well-orders A.

(6) Prove that if a theory T in a countable language is $\aleph_0$-categorical, then it is complete.

(7) Sketch the proof that the theory of dense linearly ordered sets is $\aleph_0$-categorical, and give an example showing that it is not $2^{\aleph_0}$-categorical.

(8) Let $\mathcal{L}$ be the first-order language consisting of unary predicate symbols U_n for $n \in \omega$. Let T be the theory axiomatized by:
- $\exists^{\geq n} x(\neg U_0 x)$ for all $n \in \omega$.
- $\exists^{\geq n} x(U_n x \wedge \neg U_{n+1} x)$ for all $n \in \omega$.
- $\forall x(U_{n+1} x \to U_n x)$ for all $n \in \omega$.

Describe all of the countable models of T. Which is saturated? Which is atomic? Justify your answers. (You may assume that T is complete.)

(9) Prove that the theory of the additive ordered reals $(\mathbb{R}, +, <)$ does not have a countable saturated model.

(10) Let $\mathcal{L}$ be the language unary predicates U_n for $n \in \omega$. Let T be the theory axiomatized by the scheme:

$$\exists^{\geq n} x(\bigwedge_{i \in I} U_i x \wedge \bigwedge_{j \in J} \neg U_j x)$$

for all $n \in \omega$ and for all pairs of finite disjoint sets $I, J \subseteq \omega$. Show that T has no atomic model. (You may assume that T is complete.)

Part 2

Do **FOUR** of the following problems.

(1) Assuming that ZFC is consistent, construct a Turing machine T that does not halt, yet ZFC does not prove that $\ulcorner T \urcorner$ does not terminate.

(2) Let $\pi : \omega \longrightarrow V_\omega$ be Ackermann's isomorphism. Show that there is no recursive set $R \subseteq \omega$ such that $\{\pi^{-1}(\sigma) \mid A_E \vdash \sigma\} \subseteq R$ and $\{\pi^{-1}(\sigma) \mid A_E \vdash \neg\sigma\} \subseteq \omega \setminus R$.

(3) How many consistent, complete extensions of A_E are there? Prove that your answer is correct.

(4) Let κ be an infinite cardinal. Recall that H_{κ^+} is the collection of all sets x whose transitive closure has cardinality at most κ. Define a *code* to be a pair $\langle R, \xi \rangle$ such that $R \subseteq \kappa \times \kappa$, $\langle \kappa, R \rangle$ is an extensional, well-founded structure and $\xi < \kappa$. Given a code $c = \langle R, \xi \rangle$, let $f_c : \langle \kappa, R \rangle \longrightarrow \langle u_c, \in\restriction u \rangle$ be the Mostowski-isomorphism. Say that the set coded by c is $f_c(\xi)$.
  (a) Show that if c is a code, then the set coded by c is a member of H_{κ^+}.
  (b) Show that for every member of H_{κ^+} is coded by some code.
  (c) Conclude that the cardinality of H_{κ^+} is 2^κ.

(5) Let $\langle \aleph_\alpha \mid \alpha \in \mathrm{On} \rangle$ be the monotone enumeration of the infinite cardinals.
  (a) Assume that κ is a regular limit cardinal. Show that $\kappa = \aleph_\kappa$.
  (b) Prove in ZFC that there is a club class of cardinals λ such that $\lambda = \aleph_\lambda$.

(6) Let κ be a regular uncountable cardinal. If $f : \kappa \longrightarrow \kappa$, then let's say that an ordinal $\alpha < \kappa$ is *weakly f-closed* if for all $\beta < \alpha$, $f(\beta) \leq \alpha$. Show that a set $C \subseteq \kappa$ is club in κ iff there is an $f : \kappa \longrightarrow \kappa$ such that, letting $C_f = \{\alpha < \kappa \mid \alpha \text{ is weakly } f\text{-closed}\}$, we have that
$$C \setminus \{0\} = C_f \setminus \{0\}.$$

(7) Let κ be a regular uncountable cardinal. If $f : \kappa \longrightarrow \kappa$, then let's say that an ordinal $\alpha < \kappa$ is *f-closed* if for all $\beta < \alpha$, $f(\beta) < \alpha$.
  (a) Show that if $C \subseteq \kappa$ is club, then there is an $f : \kappa \longrightarrow \kappa$ such that
$$\{\alpha < \kappa \mid \alpha > 0 \text{ and } \alpha \text{ is } f\text{-closed}\} \subseteq C.$$
  (b) Conclude that a set $S \subseteq \kappa$ is stationary iff for every $f : \kappa \longrightarrow \kappa$, there is a nonzero $\alpha < \kappa$ in S that is f-closed.

(8) For a limit ordinal λ, define
$$\mathcal{C}_\lambda = \{A \subseteq \lambda \mid \exists C \subseteq \lambda \quad C \text{ is club in } \lambda \text{ and } C \subseteq A\}.$$
Show that the following are equivalent:
  (a) $\mathcal{C}_\lambda$ is a filter of subsets of λ.
  (b) λ has uncountable cofinality.