Logic Qualifying Exam
Three Parts
December 2020

Part Zero

Provide an attribution/name and a sketch of a proof for any **THREE** of the following results:

1. If $\mathcal{A}$ is an infinite L-structure, for some first order language L, then for every cardinal κ greater than the cardinality of $\mathcal{A}$, there is an L-structure of size κ that has the same theory as $\mathcal{A}$.

2. Any two countable dense linear orders with endpoints are isomorphic.

3. If every finite subset of a collection of sentences in a fixed first order language is consistent, then the whole collection is consistent.

4. Let $(\mathcal{A}_i \mid i < \lambda)$ be a sequence of L-structures, for some fixed first order language L, such that for $i < j < \lambda$, there is an elementary embedding $\pi_{i,j} : \mathcal{A}_i \to \mathcal{A}_j$. Suppose further that these embeddings commute, that is, for $i < j < k < \lambda$, $\pi_{i,k} = \pi_{j,k} \circ \pi_{i,j}$. Then there are an L-structure $\mathcal{A}_\lambda$ and elementary embeddings $\pi_i : \mathcal{A}_i \to \mathcal{A}_\lambda$ (for $i < \lambda$) such that for $i < j < \lambda$, $\pi_i = \pi_j \circ \pi_{i,j}$ and such that the universe of $\mathcal{A}_\lambda$ is $\bigcup_{i<\lambda} \mathrm{ran}(\pi_i)$.

5. Let I be some nonempty set, and for every $i \in I$, let M_i be a structure of some fixed first order language L. Let U be an ultrafilter on I. Then for any L-formula $\varphi(x_0, \ldots, x_{n-1})$ and any $a_0, \ldots, a_{n-1} \in \prod_{i \in I} M_i$,

$$(\prod_{i \in I} M_i)/U \models \varphi[[a_0]_U, \ldots, [a_{n-1}]_U] \text{ iff } \{i \in I \mid M_i \models \varphi[a_0(i), \ldots, a_{n-1}(i)]\} \in U$$

where $[a_i]_U$ is the equivalence class of a_i modulo U.

6. If a is a set and f is a function with domain a such that for every $x \in a$, $f(x) \subseteq a$, then there is a subset of a that's not in the range of f.

7. *Alternatively:* Suppose $I \neq \emptyset$, and F is a collection of subsets of I such that if $0 < n < \omega$ and $A_0, \ldots, A_{n-1} \in I$, then $A_0 \cap \ldots \cap A_{n-1} \neq \emptyset$. Then there is an ultrafilter U on I with $F \subseteq U$.

Part One

Do FOUR of the following eight problems.

All syntax below is assumed to be first order, with equality, $=$, a logical symbol (hence always part of any of the languages considered).

1. Give a proof of the equivalence of the following three conditions, where M is a structure in a first order language L and φ is any L-formula in n free variables.

   (a) $\varphi(M)$ is finite.

   (b) $\varphi(M) = \varphi(N)$, for every elementary extension N of M.

   (c) $\varphi(N) \subseteq M^n$, for every elementary extension N of M.

2. Show that a substructure of a nontrivial dense linear ordering without endpoints is elementary if and only if it is a dense linear ordering without endpoints

3. Suppose a theory T is axiomatized by $\forall\exists$-sentences. Prove that if a structure N is the union of a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of models of T, then N is a model of T. [Here '$\subseteq$' stands for 'substructure'.]

4. Let N be the standard model of arithmetic and U be an ultrafilter on ω that contains every cofinite set of natural numbers. Prove that N^ω/U contains an element with infinitely many prime divisors.

5. Let $\mathcal{K}$ be an axiomatizable class of L-structures. Assume, for simplicity, that L has no function symbols.

   Suppose M is a structure every finite substructure of which is embeddable in some member of $\mathcal{K}$. Prove that M is embeddable in a member of $\mathcal{K}$.

6. Suppose L_E is a language whose single non-logical symbol is a binary relation symbol E. Consider the L_E-theory T of all L_E-structures in which E defines an equivalence relation with three E-classes, one of which has 1 element, one has 2, and one is infinite.

   (a) Write down an L_E-axiomatization of T.

   (b) How many non-isomorphic models does T have in each cardinality?

   (c) Conclude what you can from (b) about the completeness of T.

   (d) Is it true that, for any two models of T, one of them is elementarily embeddable in the other? [Justify your answer.]

7. Prove that a theory with quantifier elimination in a language without constant symbols is complete. [Recall: we used sentences $\perp$ for 'false' and $\top$ for 'true' and declared them both quantifier-free.]

8. Let $L_=$ be the the language with no non-logical symbols (the language of 'pure identity'). Let $T_=$ be the theory of all $L_=$-structures (i.e., the theory of all 'pure sets').

   Show that $T_=$ does not admit quantifier elimination, but all of its completions do.

Part Two

Do FOUR of the following eight problems.

1. Let A be any computable subset of $\mathrm{Th}(\mathcal{N})$, the set of sentences in the language of arithmetic which hold of the natural numbers. Let $\mathrm{Con}(A)$ be the sentence saying that there is no proof of the sentence $0 \neq 0$ from the axiom set A. Show that there exists a structure $\mathcal{M}$ which is a model of $A \cup \{\neg\mathrm{Con}(A)\}$. (Thus $\mathcal{M}$ is a model of A, yet believes that A is inconsistent.)

2. $\mathcal{N}_S = (\omega, 0, S)$ is the structure of the natural numbers under the successor function. Fix any decidable subset A of the theory of $\mathcal{N}_S$ such that every model of A consists of a copy of $\mathcal{N}_S$ along with an arbitrary number of "$\mathbb{Z}$-chains." Prove that the theory of $\mathcal{N}_S$ is decidable. (Suggestion: apply Vaught's Test to $\mathrm{Cn}(A)$, the set of all consequences of A.)

3. Assume that $\Gamma \vdash \varphi$, and that P is a relation symbol which does not occur in φ nor in any formula of Γ. Prove that there exists a deduction of φ from Γ such that P does not appear in any formula of the deduction.

4. Let $\mathcal{C}$ be a set of sets. $\mathcal{C}$ is said to be of *finite character* provided: $S \in \mathcal{C}$ iff $F \in \mathcal{C}$ for every finite $F \subseteq S$. Tukey's Lemma says that every set of finite character has a maximal element under $\subseteq$. Show (in **ZF**) that Zorn's Lemma implies Tukey's Lemma.

5. In the standard indexing of c.e. sets, W_e denotes the c.e. set which is the domain of the e-th Turing program. At which level of the arithmetic hierarchy does each of the following sets appear? As partial justification for your answers, write out an exact definition of each set, using as few quantifiers as possible. (You need not show that it is impossible to use fewer.) Here K represents the Halting Problem, or any other Σ_1^0-complete set you prefer.

   (a) $A_1 = \{e \ : \ W_e \text{ contains no element of } K\}$.
   (b) $A_2 = \{e \ : \ W_e \text{ contains every element of } K\}$.
   (c) $A_3 = \{e \ : \ W_e \text{ is infinite}\}$.
   (d) $A_4 = \{e \ : \ W_e \text{ is cofinite, i.e., } (\omega - W_e) \text{ is finite}\}$.

6. Let $(A, <)$ and $(B, \prec)$ be well-orders of two sets A and B. Without using the Axiom of Choice, prove that there exists an injective, order-preserving map either from A into B, or from B into A.

7. Show that the set $\mathrm{Fin} = \{e \ : \ W_e \text{ is finite}\}$ is a Σ_2-complete set with respect to 1-reducibility $\leq_1$. (The hard part is showing completeness, but be sure also to show that Fin is Σ_2.)

8. Let $A \subseteq \omega$ be an infinite set. Write out a reasonably detailed description of an oracle Turing program such that with A as oracle, the program computes the function outputting the least element of A greater than x:

$$f(x) = \mu y(y > x \text{ and } y \in A).$$

   Next, show that there is no oracle Turing program which, given any (finite or infinite) oracle B, computes the function

$$g(x) = \begin{cases} \mu y(y > x \text{ and } y \in B) & \text{if there exists such a } y \\ 0 & \text{if not.} \end{cases}$$