
Logic Qualifying Exam
Three Parts
August 17, 2021

Part Zero (16 points)

Answer each of the following questions fully. None should take more than a few paragraphs, and some may need less than that. It is not advisable to spend more than 30 minutes on Part Zero.

1. Explain why the Completeness Theorem implies the Compactness Theorem for first-order languages.
2. Give the name of the theorem that states that every infinite finite-branching tree has an infinite path, and explain briefly (in **ZFC**) how to find such a path in the tree. (A tree is *finite-branching* if every element has only finitely many immediate successors. A *path*, here, is any linearly ordered subset of the tree.)
3. Let $T_1 \subsetneq T_2 \subsetneq \dots$ be a strictly increasing sequence of theories in a finite language. Show that the union $T = \bigcup_n T_n$ is a consistent theory which has an infinite model and is not finitely axiomatizable. (Here *theory* denotes a set of sentences which is closed under deduction. The set may or may not be consistent, and may or may not be complete.)
4. (a) Explain how, from the Löwenheim-Skolem Theorem and the Completeness Theorem, one can quickly deduce the Los-Vaught Test. (The point is to sketch the proof. You need to know these theorems in order to do so, but you need not state the theorems themselves.)
(b) The Los-Vaught Test includes the assumption that the theory in question has no finite models. Show that this assumption is necessary, by describing an incomplete theory that is categorical in some infinite power.
5. Suppose that $(A, <)$ and $(B, <)$ are well orderings. Suppose further that $f_0 : A \rightarrow B$ and $f_1 : A \rightarrow B$ are order-preserving embeddings of A into B , and that each of $\text{ran}(f_0)$ and $\text{ran}(f_1)$ forms an initial segment of $(B, <)$. Show that $f_0 = f_1$.
6. Intuitively, a computably enumerable set is the range of a partial computable function. However, the standard definition defines it to be the domain of a partial computable function. Sketch a proof that these are equivalent definitions.

Part One (42 points)

Do THREE of the following six problems (which continue on the next page). All syntax is first order—with equality, $=$, a logical symbol (hence always part of any of the languages L considered). Please justify your answers with full proofs, where you may quote well-known results by name (without proof).

- As usual, consider vector spaces over a field K in a language with 0 and $+$ and a unary function symbol k for every scalar $k \in K$.
 - Prove that every infinite-dimensional subspace of a vector space is an elementary substructure.
 - Derive from this that the theory of infinite vector spaces over a finite field is complete. (Note: this is true for arbitrary fields, but somewhat harder to prove.)
- Let L_P be the language whose only non-logical symbol is a unary predicate P . Let T_P be the L_P -theory axiomatized by the single axiom $\forall x P(x) \vee \forall x \neg P(x)$. Let T_P^∞ be the theory of all infinite models of T_P .
 - Axiomatize T_P^∞ .
 - Prove that, given any joint non-empty substructure A of two models M and N of T_P^∞ , the expanded structures (M, A) and (N, A) are elementarily equivalent—which means that M and N satisfy the same $L_P(A)$ -sentences when every name for $a \in A$ is interpreted in M or N by the actual element a itself.
 - Show that nevertheless, T_P^∞ does not have quantifier elimination. (Hint: one may use a general result here about languages without constant symbols or else argue directly by considering an appropriate formula.)
- Suppose L_E is a language whose single non-logical symbol is a binary relation symbol E . Consider the L_E -theory T of all L_E -structures in which E defines an equivalence relation with infinitely many finite E -classes, one of each finite nonzero cardinality.
 - Write down an L_E -axiomatization of T .
 - Show that T has a model with infinitely many infinite E -classes. (Hint: as warm-up you may want to show first that T has a model with an infinite E -class.)
 - Conclude what you can about categoricity or non-categoricity of T .
- Consider the theory T from the previous problem.
 - Show that, given two models M and N of T which have the same number of infinite classes and such that $M \subseteq N$, then $M \preceq N$.
 - Using the previous, prove that, given any two models M and N of T with $M \subseteq N$, we have $M \preceq N$.
- Suppose T is an L -theory, φ is an L -formula with one free variable and c is a constant symbol not contained in L . Let \models_L be the entailment (consequence) relation in L , and $\models_{L(c)}$ the same relation in the expansion $L(c)$ of L by the new constant symbol c .
Prove that $T \models_{L(c)} \varphi(c)$ implies $T \models_L \forall x \varphi(x)$. (Mind the details!)

6. Let $D(M)$ be the (quantifier-free) diagram of the L -structure M . Let $L(M)$ be the expansion of L by constant symbols (names) for every element of M .

Prove that there is a monomorphism from M into an L -structure N if and only if N has an $L(M)$ -expansion which is a model of $D(M)$.

Part Two (42 points)

Do **THREE** of the following six problems.

1. Show that a set $A \subseteq \omega$ is computable if and only if A can be recursively enumerated in strictly increasing order.
2. Work with Turing machines over the fixed alphabet $\Sigma = \{0, 1\}$. For such a Turing machine M , let $\langle M \rangle$ be a (canonical) string coding M , and let $L(M)$, the language of M , be the set of words accepted by M (that is, the strings on which M halts). Show that the language

$$\{\langle M \rangle \mid (\forall \text{ words } w) M \text{ accepts } w \text{ iff it accepts the word } w \text{ written backwards}\}$$

is undecidable.

3. Working in **ZFC**, suppose $\varphi(x, y)$ is a Σ_1 -formula in the language of set theory. Let $\alpha \geq \omega$ be an ordinal, and suppose that β , an ordinal, is unique such that $\varphi(\beta, \alpha)$ holds. Show that $\beta < \alpha^+$ (where α^+ is the least cardinal greater than α).
4. Show (in **ZF**): if κ is an uncountable regular cardinal and $S \subseteq \kappa$ is stationary in κ , then there is a set $\bar{S} \subseteq S$, stationary in κ , such that whenever $\alpha \in \bar{S}$, then $\bar{S} \cap \alpha$ is not stationary in α .
Hint: Define whether $\alpha \in \bar{S}$ or not by recursion on α in the obvious way.
5. Let \mathcal{N} be the standard model of arithmetic, that is, $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot \rangle$, and let \mathcal{L} be its language. Show (in **ZFC**) that there is a collection \mathcal{I} of pairwise non-isomorphic countable \mathcal{L} -structures, each a model of $\text{Th}(\mathcal{N})$, such that \mathcal{I} has cardinality 2^ω , and show that there is no such \mathcal{I} of greater cardinality.

Hint: You can argue (but then also have to prove) that for any set A of prime numbers, there is a countable model of $\text{Th}(\mathcal{N})$ containing an element whose (standard) prime factors are exactly the members of A .

6. Work in **ZFC**, and assume **ZFC** is consistent. Recall that a cardinal κ is inaccessible iff it is regular, uncountable, and has the property that for every cardinal $\alpha < \kappa$, $2^\alpha < \kappa$.
 - (a) Show that inaccessible cardinals are limit cardinals.
 - (b) Show that **ZFC** does not prove that there is an inaccessible cardinal, by showing that **ZFC** proves that if there is an inaccessible cardinal, then there is a transitive model of **ZFC** with no inaccessible cardinal.
 - (c) Prove (in **ZFC**, say) that $\text{con}(\mathbf{ZFC})$ does not imply $\text{con}(\mathbf{ZFC} + \text{“there is an inaccessible cardinal”})$.