Logic Comprehensive Exam
Spring 2015

Part Zero

Answer both of the following questions.

1. Formulate precisely the Completeness Theorem and the Compactness Theorem, and outline the proof of one of them.

2. Formulate precisely Gödel's First and Second Incompleteness Theorems, and outline the proof of one of them.

Part One

Do four of the following eight problems.

1. Let T be an inductive theory (i.e. T is preserved under unions of chains). Show that if $\mathfrak{M} \models T$ then there is $\mathfrak{N} \models T$ so that $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{N}$ is existentially closed.

2. Let $\mathfrak{M}$ be a first order structure and let $\mathfrak{M}_i$ be copies of $\mathfrak{M}$ indexed by a set I. Let $\mathcal{U}$ be an ultrafilter on I and let $\mathfrak{N} = (\Pi_i \mathfrak{M}_i)/\mathcal{U}$. Show that the map $f : \mathfrak{M} \to \mathfrak{N}$ defined by $f(a) = (a_i)_{i \in I}$ is elementary.

3. Let T be a first order theory and suppose that whenever $\mathfrak{M} \subseteq \mathfrak{N}$ are models of T then any existential formula with parameters from M true in $\mathfrak{N}$ is also true in $\mathfrak{M}$. Show that T is model complete.

4. Let T be the first order theory in a language with two binary relation symbols $<_0$ and $<_1$ axiomatized as follows:

   - "$<_0$ is a dense linear order without endpoints";
   - "$<_1$ is a dense linear order without endpoints";
   - $\forall x_0, y_0 \forall x_1, y_1((x_0 <_0 y_0 \wedge x_1 <_1 y_1) \to \exists z(x_0 <_0 z <_0 y_0 \wedge x_1 <_1 z <_1 y_1))$.

   You may assume that T is consistent. Show that T is ω-categorical.

5. Let K be an algebraically closed field of characteristic 0 and let $A \subseteq K$. Show that if two elements $a, b \in K$ both are not in the smallest algebraically closed subfield of K containing A then $tp(a/A) = tp(b/A)$.

6. Let T be a theory in a countable language containing at least one constant c and one unary function symbol f. Suppose that for any formula with one free variable $\varphi(x)$ and any model $\mathfrak{M} \models T$ there is $n \in \mathbb{N}$ so that $\mathfrak{M} \models \varphi(f^n(c))$. Show that there is a model $\mathfrak{N} \models T$ so that $N = \{f^n(c) : n \in \mathbb{N}\}$.

7. Let T be any theory in a language containing a binary relation symbol $<$. Suppose that the axioms of T imply that $<$ is a dense linear ordering without endpoints. Suppose further that in no model $\mathfrak{M}$ of T is there an infinite definable set X so that for each $x \in X$ there are $a, b \in M$ so that $(a, b) \cap X = \{x\}$ (where (a,b) is the open $<$ interval between a and b). Show that for any formula $\varphi(x, \overline{y})$ there is $N \in \mathbb{N}$ so that if $\mathfrak{M} \models T$ and $\overline{a} \in M^{|\overline{y}|}$ and $\varphi(M, \overline{a})$ has cardinality greater than N then $\varphi(M, \overline{a})$ is infinite.

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8. Let T be complete theory in a countable language with no finite models. Let $\mathfrak{M}$ be a countable model of T so that $\mathfrak{M}$ is countably universal (i.e. every countable model of T elementarily embeds in $\mathfrak{M}$) and countably homogeneous (i.e. for any $\bar{a}$ and $\bar{b}$ in M if $tp(\bar{a}) = tp(\bar{b})$ then there is an automorphism of $\mathfrak{M}$ sending $\bar{a}$ to $\bar{b}$. Show that $\mathfrak{M}$ is ω-saturated.

Part Two

Do four of the following eight problems. A_E refers to the axiom set given in Enderton for the structure $\mathcal{N} = (\omega, 0, S, <, +, \cdot, E)$, and $\langle\psi\rangle$ denotes the Gödel number of a formula ψ.

1. Show that there is no decidable set C such that, for all formulas $\psi \in \mathbf{Cn}(A_E)$:

$$\langle\psi\rangle \in C \quad \& \quad \langle\neg\psi\rangle \notin C.$$

2. Prove the Generalization Theorem: if no well-formed formula in a set Γ contains x as a free variable, and if $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall x\varphi$. (For the method of deduction defining the symbol $\vdash$, you may use the definition from Enderton; you may also use any other equally powerful notion, but if so, please explain it as you use it.)

3. Let A be a (not necessarily decidable) set of formulas in the language of arithmetic. Describe in reasonable detail a Turing reduction from the set $\{\langle\varphi\rangle : A \vdash \varphi\}$ of Gödel numbers of $\mathbf{Cn}(A)$ to the jump A' of A.

4. Show that the set $\{e : |W_e| \geq 3\}$ is computably enumerable, but not computable. (For noncomputability, use of the Recursion Theorem is suggested, but not required.)

5. Prove that a set $S \subseteq \omega$ is Δ^0_2 if and only if the characteristic function of S is the limit of a computable sequence of computable functions. (Recall that being Δ^0_2 means that S is both Σ^0_2 and Π^0_2.)

6. Show that the jump A' of an arbitrary set $A \subseteq \omega$ satisfies $A <_T A'$. Be sure to prove both the Turing reducibility and the strictness of the inequality.

7. Show that, under the axiom system **ZF**, the following two versions of the Axiom of Choice are equivalent.

   (AC1): For each set R of ordered pairs, there is a function $H \subseteq R$ with $\text{dom}(H) = \text{dom}(R)$.

   (AC2): For every set I and every function f with domain I and with $\emptyset \notin \text{ran}(f)$, there exists a function g with domain I such that

$$(\forall i \in I)\ g(i) \in f(i).$$

   (AC2 is somtimes stated as "the Cartesian product $\prod_{i \in I} X_i$ of nonempty sets is nonempty," with X_i denoting $f(i)$. The function g described above is viewed as an element of this Cartesian product, with i-th coordinate $g(i)$.)

8. Do there exist nonzero ordinals α, β, γ, and δ such that under ordinal addition:

   - $\alpha + \omega = \alpha$?
   - $\omega + \beta = \beta$?
   - $\gamma + \omega = \omega$?
   - $\omega + \delta = \omega$?

   Answer each of these four questions separately, and prove each answer.