
Logic Qualifying Exam
Spring 2016

Part Zero

Answer all of the following questions.

1. Let \mathcal{L} be a first order language without constant or function symbols. Suppose that \mathfrak{M} and \mathfrak{N} are \mathcal{L} -structures and that $f : M \rightarrow N$ is an isomorphism between \mathfrak{M} and \mathfrak{N} . Show that if $\varphi(x_1 \dots x_n)$ is an \mathcal{L} -formula and $a_1 \dots a_n \in M$ then $\mathfrak{M} \models \varphi(a_1, \dots, a_n)$ if and only if $\mathfrak{N} \models \varphi(f(a_1), \dots, f(a_n))$.
2. Let $\mathcal{L} = \{+, \times, <, 0, 1\}$ and let $\mathfrak{R} = \langle \mathbb{R}, +, \times, <, 0, 1 \rangle$ (Here the symbols are interpreted in the obvious way). Show that there is a model \mathfrak{M} of $\text{Th}(\mathfrak{R})$ and an element $m \in M$ so that $\mathfrak{M} \models 0 < m$ and $\mathfrak{M} \models n \times m < 1$ for each $n \in \mathbb{N}$.

3. Show that the Soundness and Completeness Theorems directly imply the Generalization Theorem:

For a set Γ of formulas and a formula φ , if $\Gamma \vdash \varphi$ and no formula in Γ has x free, then $\Gamma \vdash \forall x\varphi$.

(You should not need to use any form of induction on formulas or on length of deductions.)

4. The theory ACF_0 of algebraically closed fields of characteristic 0 is complete and decidable, and has a computable axiomatization.
 - (a) Explain why this does not contradict the First Incompleteness Theorem.
 - (b) Is there a proof from ACF_0 of the consistency of ACF_0 ?

Part One

Do four of the following eight problems.

1. Using Tarski-Seidenberg—that is to say, quantifier elimination for the real ordered field in the signature $\langle 0, 1; +, -, \cdot, < \rangle$ —show that the field of real numbers viewed as a pure field—that is to say, in the signature $\langle 0, 1; +, -, \cdot \rangle$ —is model-complete.
2. Let L be a language and T an L -theory. Let σ_n and τ_n , for $n \in \mathbb{N}$, be sentences in this language. Show that if the infinitary conjunction $\bigwedge_{n \in \mathbb{N}} \sigma_n$ is equivalent modulo T to the infinitary disjunction $\bigvee_{n \in \mathbb{N}} \tau_n$, then these two infinitary sentences are actually equivalent modulo T to a (first-order) sentence, as follows. For each n , let T_n be the theory consisting of T , all σ_i for $i \in \mathbb{N}$, and $\neg\tau_1, \dots, \neg\tau_n$. Show that some T_n must be inconsistent, by using an ultraproduct construction or appealing to the Compactness/Finiteness Theorem. Now deduce that the infinitary sentence $\bigvee_{n \in \mathbb{N}} \tau_n$ is first-order modulo T .
3. Prove that if $(\mathcal{M}_i | i \in \mathbb{N})$ is an elementary chain of L -structures, in the sense that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ for each i , and if \mathcal{M} is the union of this chain, then $\mathcal{M}_0 \preceq \mathcal{M}$.
4. Let \mathbf{S}_L be the Stone space of L consisting of all complete L -theories T . Show that a theory T is finitely axiomatizable if and only if it is an isolated point of \mathbf{S}_L . (Recall that a point x in a topological space S is called *isolated* if the singleton $\{x\}$ is open.)
5. Give two non-isomorphic algebraically closed fields that have some nontrivial ultrapowers which are isomorphic. Substantiate all your claims about these fields.
6. Express the following statements in first-order logic.
 - (a) In the empty language, for a positive integer p : “there are at most p different elements.”
 - (b) In the language of rings $\{+, \cdot, 0, 1\}$: “the characteristic is p .”
 - (c) In the language of graphs $\{R\}$: “there is a path of length exactly 3 (i.e., a path with exactly 3 edges) between x and y .”
 - (d) In the language of orderings $\{<\}$: “every element has an immediate predecessor.”
7. Let S and T be L -theories such that $S_\forall \subseteq T \subseteq S$. Show that if T admits elimination of quantifiers, then $\text{Mod}(T) = \text{Mod}(S)$.
8. Let \mathcal{M} be an L -structure. We define a category $\mathbb{D}_n(\mathcal{M})$ in which the objects are the definable (with parameters) subsets of M^n ; a *morphism* $F: X \rightarrow Y$ of definable subsets $X, Y \subseteq M^n$ is then given by a (model-theoretic) homomorphism $f: \mathcal{M} \rightarrow \mathcal{M}$ such that its n -th Cartesian power $f^n: M^n \rightarrow M^n: (a_1, \dots, a_n) \mapsto (f(a_1), \dots, f(a_n))$ maps X inside Y , in which case F is then the restriction of f^n to X . Show that the projection map $\mathbb{D}_n(\mathcal{M}) \rightarrow \mathbb{D}_{n-1}(\mathcal{M})$ sending a definable subset $|\varphi(x_1, \dots, x_n)|$, with $\varphi(x_1, \dots, x_n) \in L_n(\mathcal{M})$, to the definable subset $|\langle \exists x_n \rangle \varphi(x_1, \dots, x_n)|$, can be made into a functor.

Part Two

Do four of the following eight problems.

A_E refers to the finite axiom set given in Enderton for the structure $\mathcal{N} = (\omega, 0, S, <, +, \cdot, E)$.

1. Pressburger arithmetic is the structure $\mathcal{N}_A = (\mathbb{N}, 0, S, <, +)$. Show that the theory of \mathcal{N}_A does not admit quantifier elimination, by giving a formula $\theta(x)$ in this language and proving that there is no quantifier-free formula $\psi(x)$ equivalent to it under $\text{Th}(\mathcal{N}_A)$.
2. Prove the strong undecidability of the set of consequences of the axiom set A_E : every theory T in the language of arithmetic consistent with A_E is undecidable.
(You may not cite the theorem which says that this is true, but you may refer to earlier results if you like.)
3. Prove that the set of ordered pairs (a, b) of consecutive prime numbers in ω is representable in $\text{Cn}(A_E)$, the closure of A_E under deducibility.
4. It was shown by Julia Robinson in 1949 that in the structure $\Omega = (\mathbb{Q}, +, \cdot, 0, 1)$, there is a formula $\zeta(x)$ in one free variable x which defines the subset \mathbb{Z} . Use this fact to show that the theory of Ω is undecidable.
5. Prove that each of the two sets

$$\text{Inf} = \{e \in \mathbb{N} : W_e \text{ is infinite}\} \quad \text{Tot} = \{n \in \mathbb{N} : W_n = \mathbb{N}\}$$

is m -reducible to the other. (In fact these sets are both Π_2 -complete; but you need to give a direct proof, without citing Π_2 -completeness.)

6. Suppose that φ is a partial computable function whose domain is not computable. Prove that there exists an $n \in \omega$ such that in \mathcal{N} , φ does not halt on input n , but such that there exists a model of A_E in which φ does halt on input n .
7. A *Cantor normal form* of an ordinal α is a finite nonincreasing sequence $\beta_0 \geq \beta_1 \geq \dots \geq \beta_n$ of ordinals such that

$$\alpha = \omega^{\beta_0} + \omega^{\beta_1} + \dots + \omega^{\beta_n}.$$

Show that for every ordinal α , the Cantor normal form of α (if one exists) must be unique. (In fact it does exist for every α , but you need not prove existence.)

8. Let $(A, <)$ and $(B, <)$ be arbitrary well-orders of two sets A and B . Prove that one of these must be order-isomorphic to some (not necessarily proper) initial segment of the other. Give an explicit definition of the isomorphism from the one to the initial segment of the other.