Logic Qualifying Exam
Three Parts
Spring 2017

Part Zero

Answer all of the following questions.

1. State Gödel's First and Second Incompleteness Theorems correctly, and sketch the proof of one of them. (A paragraph or so is sufficient, if written accurately.)

2. Explain how, from the Löwenheim-Skolem Theorem and the Completeness Theorem, one can quickly deduce the Łos-Vaught Test, which states that if κ is infinite and T is a κ-categorical theory with no finite models, in a language with at most κ-many symbols, then T is complete.

3. State the Compactness Theorem (also known as the Finiteness Theorem) correctly, and use it to sketch a proof that, for linear orders (in the language with just $<$ and equality), the property of being a well-order is not first-order definable.

4. Define the Halting Problem, and explain why it is undecidable.

Part One

Do four of the following seven problems.

1. Let $S := \{E\}$ be the signature with one binary relation symbol E. For $n \in \mathbb{N}$, let T_n be the S-theory saying that E is symmetric and irreflexive, and for each point p there are exactly n points q with pEq.

   (a) Write down the first-order sentences axiomatizing T_n for $n = 0, 1, 2$.

   (b) For what n is T_n complete?

2. Consider the following statements; prove the ones that are true, and give explicit examples disproving the ones that are not true.

   - If every finite substructure of $\mathcal{A}$ satisfies a sentence θ, then $\mathcal{A}$ satisfies θ.
   - Let $L := \{+, -, 0\}$ be the language of groups. Up to L-isomorphism, there are exactly two L-structures $\mathcal{A}$ with $|A| = 4$.

3. Let $S := \{P\}$ be the signature with one binary function symbol P. Let $\mathcal{A}$ be the S-structure with universe $\mathbb{Z}$ and $P^{\mathcal{A}}(a, b) := a + b$.

   (a) What substructures of $\mathcal{A}$ are elementarily equivalent to $\mathcal{A}$?

   (b) Does $\mathcal{A}$ have any proper elementary substructures? I.e., is $\mathcal{A}$ *minimal*?

4. Let S and $\mathcal{A}$ be as in the previous problem. Show that for any S-structure $\mathcal{B}$ that is elementarily equivalent to $\mathcal{A}$, there is an elementary embedding from $\mathcal{A}$ to $\mathcal{B}$; or give a counterexample. That is, determine whether $\mathcal{A}$ is *prime*.

5. Prove that one of the following properties is equivalent to the conjunction of the other two.

   (a) A countable S-structure $\mathcal{M}$ is *homogeneous* if every partial S-elementary function between finite subsets of M extends to an automorphism of $\mathcal{M}$.

   (b) A countable S-structure $\mathcal{M}$ is *universal* if for every countable S-structure $\mathcal{N}$ that is elementarily equivalent to $\mathcal{M}$, there exists an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$.

   (c) A countable S-structure $\mathcal{M}$ is *saturated* if for every finite subset $A \subset M$ and every consistent $S(A)$-type $p(x)$, there is an element $b \in M$ that realizes p.

6. Let X be a set of S-sentences. Recall that $Mod(X)$ is the class of all S-structures satisfying all sentences in X. Recall that for a class K of S-structures, $Th(K)$ is the set of all S-sentences satisfied by all S-structures in K. Prove that one of the following properties is equivalent to the conjunction of the other two.

   (a) X is complete.
   That is, for every S-sentence θ, at least one of θ and $\neg\theta$ is in X.

   (b) $Th(Mod(X)) = X$

   (c) $Th(Mod(X))$ is complete.

7. Use a Henkin construction to prove the Omitting Type Theorem:
   Fix a signature S, suppose T is a set of S-sentences, $p(x)$ is a set of S-formulae with one free variable x, and $T \cup p(x)$ is satisfiable. Suppose also that p is *not isolated modulo* T, i.e., no

finite conjunction of formulae in p implies all formulae in p modulo T: i.e., there do not exist $\varphi_i(x) \in p(x)$ such that for all $\psi(x) \in p(x)$

$$T \models (\wedge_{i=1}^{i=n} \varphi_i(x)) \to \psi(x)$$

Show that there is an S-structure $M \models T$ such that no element of M realizes p, i.e., for all $a \in M$ there is a formula $\psi(x) \in p(x)$ such that $M \models \neg\psi(a)$.

Don't try to do this with Compactness - it won't work.

Part Two appears on the following page.

Part Two

Complete four of the following eight problems.

1. Assume that PA is consistent. Prove that there is a Turing machine program p which does not halt on input 0, but such that PA cannot prove that p fails to halt on input 0.

2. Prove that every finitely axiomatizable complete theory is decidable.

3. Assume PA is consistent. True or false: there is a consistent theory T extending PA that proves its own inconsistency $T \vdash \neg\text{Con}(T)$. Prove your answer.

4. True or false: there is a universal total computable function, that is, a total computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that every total computable function occurs as f_n for some n, where f_n is the function defined by $f_n(m) = f(n, m)$. Prove your answer.

5. Prove that the collection of Turing machine programs p which halt on only finitely many inputs is not decidable.

6. Show that the infinite-sum operation $\oplus_n A_n = \{\langle n, k\rangle \mid k \in A_n\}$, where $\langle n, k\rangle$ is any of the usual computable pairing functions on the natural numbers, is not well-defined with respect to Turing degree. That is, there are $A_n \equiv_T B_n$, with $\oplus_n A_n \not\equiv_T \oplus_n B_n$.

7. Consider the structure $\langle \mathbb{Z}, <, U\rangle$, where U is a unary predicate on the integers $\mathbb{Z}$. Show that if U is not periodic, then each pair of distinct points in the structure have different 1-types.

8. Show that the countable random graph has no definable elements.