Logic Comprehensive Exam
Spring 2018

Three Parts

Part Zero

1. State and prove the Łos ultraproduct theorem.
2. State and prove Cantor's theorem on countable endless dense linear orders.
3. State the compactness theorem of first-order logic and sketch a proof of it.
4. Define the halting problem and explain why it is not decidable.

Part One

Do four of the following eight problems.

1. For T a first order theory let $T_\forall$ be the set of all universal sentences σ so that $T \vdash \sigma$. Show that $\mathfrak{M} \models T_\forall$ if and only if $\mathfrak{M} \subseteq \mathfrak{N}$ for some $\mathfrak{N} \models T$.

2. Let T be a complete first order theory and suppose that for every type $p(\overline{x})$ there is a set of quantifier free formulas $q(\overline{x}) \subseteq p(\overline{x})$ so that if $\mathfrak{M} \models T$, $\overline{a} \in M$, and $q(\overline{a})$ holds then $p(\overline{a})$ holds. Show that T has quantifier elimination.

3. Let $\mathcal{L} = \{U_n : n \in \omega\}$ where each U_n is a unary predicate. Let T be the theory axiomatized by:

   - $\forall x U_0(x)$;
   - $\forall x(U_{n+1}(x) \to U_n(x))$ for all $n \in \omega$;
   - $\exists x_1, \dots x_m(\bigwedge_{i=1}^m (U_n(x_i) \wedge \neg U_{n+1}(x_i)) \wedge \bigwedge_{i \neq j} x_i \neq x_j)$; for all $m \in \omega$ and all $n \in \omega$.

   Describe all of the countable models of T and indicate which is prime and which is saturated. Justify you answer.

4. Let $\mathfrak{M}_i$ be finite $\mathcal{L}$-structures so that $|M_i| \neq |M_j|$ when $i \neq j$. Let $\mathcal{U}$ be an ultrafilter on ω. Show that if the ultraproduct $\mathfrak{M} = (\Pi_{i\in\omega}\mathfrak{M}_i)/\mathcal{U}$ is finite then $\mathfrak{M} \cong \mathfrak{M}_i$ for some $i \in \omega$.

5. Let $\mathcal{L}$ be the language consisting of a single binary relation R and a single unary relation U. Let T be axiomatized by:

   - $\forall x, y(Rxy \to Ux \wedge \neg Uy)$;
   - $\forall x(Ux \to \exists y_1, \dots, y_m(\bigwedge_{i=1}^m Rxy_i \wedge \bigwedge_{i \neq j} y_i \neq y_j))$ for all $m \in \omega$;
   - $\forall x_1, \dots, x_n, y_1, \dots y_n(\bigwedge_{1 \le i,j, \le n} x_i \neq y_j \to \exists z(\bigwedge_{i=1}^n Rzx_i \wedge \bigwedge_{j=1}^n \neg Rzy_j))$.

   Show that T is ω-categorical.

6. Let $\mathcal{L}$ be the language consisting of countable many unary predicates U_i for $i \in \omega$. Let T be axiomatized by:

   - $\exists x(\bigwedge_{i \in I} U_i x \wedge \bigwedge_{j \in J} \neg U_j x)$ for any two disjoint finite I, J subsets of ω.

   Show that T does not have a countable saturated model.

7. Let T be a theory in a countable language $\mathcal{L}$. Suppose that $\mathcal{L}$ has at least countably many distinct constant symbols c_i for $i \in \omega$. Suppose that $\varphi(x)$ is a $\mathcal{L}$-formula so that for any $\mathcal{L}$-formula $\psi(x)$ so that $T \vdash \forall x(\psi(x) \to \varphi(x))$ there is $i \in \omega$ so that $T \vdash \psi(c_i)$. Show that there is a model $\mathcal{M} \models T$ so that $\varphi(M) \subseteq \{c_i^{\mathfrak{M}} : i \in \omega\}$.

8. Let $\mathcal{L}$ be the first order language with a single binary relation symbol E. Let T be the theory axiomatized by stating that E is an equivalence relation with infinitely many equivalence classes and that each equivalence class is infinite. Show that if $\varphi(x)$ is any $\mathcal{L}$-formula then there is $N \in \omega$ so that if $\mathfrak{M} \models T$ and $|\varphi(M)| > N$ then $\varphi(M)$ is infinite.

Part Two

Do four of the following eight problems.

1. How many consistent, complete extensions of A_E are there? Prove that your answer is correct.

2. Let T be a recursive, consistent set of sentences containing the axioms of ZFC. A sentence φ is *independent* of T if neither φ nor $\neg\varphi$ are provable from T. Show that the set of sentences that are independent of T is not recursively enumerable.

3. A Turing machine with finite tape is defined like a regular Turing machine, except that its tape has only finitely many cells. When during operation the read/write head is on the last cell of the tape and it is told to move right, it just stays put. Is the Halting Problem for Turing machines with finite tape decidable or undecidable (by a regular Turing machine)? Prove your answer is correct.

4. Show that there are inequivalent Turing machines M and N such that M outputs $\ulcorner N \urcorner$ on empty input and N outputs $\ulcorner M \urcorner$ on empty input.

5. Fix a finite alphabet Σ. For a word w over Σ, let w^r denote the word w, written backwards. Show that the problem $R = \{\ulcorner T \urcorner \mid \text{for all } w \in \Sigma^*, T \text{ accepts } w \text{ iff } T \text{ accepts } w^r\}$ is undecidable.

6. Let κ be an uncountable regular cardinal.

   (a) Show that the set of singular limit ordinals less than κ is a stationary subset of κ.

   (b) Assuming that κ is a strong limit cardinal, show that the set of strong limit cardinals less than κ is closed and unbounded in κ.

   (c) Conclude that the set of singular strong limit cardinals below a strongly inaccessible cardinal is stationary.

7. Let κ be a strongly inaccessible cardinal, and let $M = \langle V_\kappa, \in\restriction V_\kappa \rangle$, a model of the language of set theory. Let $N = \langle X, \in \cap (X \times X) \rangle \prec M$ be a countable elementary submodel.

   (a) Show that $N \cap X \in \omega_1$ and $\omega_1 \in X$.

   (b) Show that X is not transitive.

   (c) Show that N is extensional and well-founded.

   (d) Let $f : N \longrightarrow \langle U, \in\restriction U \rangle$ be the Mostowski collapse and isomorphism. Show that $f \restriction \omega_1$ is the identity and $f(\omega_1) = \omega_1 \cap X$.

8. Let $\kappa > \omega_1$ be a regular cardinal, and let $f : {}^\omega\kappa \longrightarrow \kappa$ be such that for every $\alpha < \kappa$, $f \restriction {}^\omega\alpha$ is bounded in κ. Say that $\alpha < \kappa$ is f-closed if for every $\beta < \alpha$ and every $s \in {}^\omega\beta$, $f(s) < \alpha$. Show that the set

$$C = \{\alpha < \kappa \mid \alpha \text{ is } f - \text{closed}\}$$

   is club in κ, and that the set of $\alpha < \kappa$ such that for every $s \in {}^\omega\alpha$, $f(s) < \alpha$ is stationary in κ.