Part Zero (16 points)

Answer each of the following questions fully. None should take more than a few paragraphs, and some may need less than that. It is not advisable to spend more than 30 minutes on Part Zero.

1. Give a detailed proof that there is no largest cardinal. Cite axioms from $\text{ZFC}$ as needed.

2. Among the following three sets, which are $m$-reducible (alternatively, 1-reducible) to which others? (This is really six questions, but your justifications may be quite brief.)

   \[ \text{Th}(\mathcal{N}) \quad \text{Cn}(PA) \quad \text{the Halting Problem } K. \]

   Here $\mathcal{N} = (\mathbb{N}, +, \cdot)$ is the structure of the natural numbers under addition and multiplication, and Cn($PA$) is the set of theorems provable from the (decidable) set $PA$, which consists of the axioms of Peano Arithmetic.

3. Let $A$ be a substructure of $B$. Suppose that for every finite tuple $a_1, \ldots, a_n$ of elements of $A$ and for every $b \in B$, there is an automorphism $f$ of $B$ with $f(a_i) = a_i$ for each $i \leq n$ and $f(b) \in A$. Prove that $A$ is an elementary substructure of $B$.

4. Prove that it is consistent with the theory of the structure $(\mathbb{N}, 0, 1, <, +, \cdot)$ for a number to have infinitely many prime factors.
Part One (42 points)

Do THREE of the following six problems (which continue on the next page). All syntax is first order—with equality, $=,$ a logical symbol (hence always part of any of the languages $L$ considered). Please justify your answers with full proofs, where you may quote well-known results by name (without proof).

1. Let $\varphi(x,y)$ be an $L$-formula in the free variables, $x$ and $y$. Given an $L$-structure $M$ and an element $c \in M$, the parametric formula $\varphi(x,c)$ defines a set in $M$, denoted by $\varphi(M,c)$.

   (a) Prove: if $T$ is a complete $L$-theory such that, for every model $M \models T$ and every element $c \in M$, the set $\varphi(M,c)$ is finite, then there is a uniform finite bound on the cardinality of all those sets $\varphi(M,c)$.

   (b) Prove or disprove the converse: when there is such a uniform bound, all the sets $\varphi(M,c)$ are finite (in all $M \models T$).

2. (a) Prove or disprove: the ordering of the rationals embeds elementarily into any non-trivial dense linear order without endpoints.

   (b) Prove or disprove: any two non-trivial dense linear order without endpoints are elementarily equivalent.

3. Let $L_P$ be the language whose only non-logical symbol is a unary predicate $P$. Let $T_P$ be the $L_P$-theory axiomatized by the single axiom $\forall x P(x) \lor \forall x \neg P(x)$. Let $T_P^\infty$ be the theory of all infinite models of $T_P$.

   (a) How to axiomatize $T_P^\infty$?

   (b) How many models, up to isomorphism, does $T_P$ have in any given cardinality (finite or infinite)?

   (c) How many completions does $T_P^\infty$ have?

   (d) How many models does each of $T_P^\infty$’s completions have in any given infinite cardinality?

   (e) Prove that every embedding of non-empty models of $T_P^\infty$ is elementary.

4. Suppose $L_g = (1, \cdot)$ with the only nonlogical symbols, a constant symbol 1 and a binary function symbol $\cdot$.

   Prove or disprove: the theory of groups is axiomatizable by universal $L_g$-sentences, i.e., $L_g$-sentences such that, when they are written in prenex normal form, the only quantifiers are $\forall$.

5. Suppose $L_Z = (0, +, -)$ with a constant, 0, and two binary function symbols, + and −.

   Consider the additive group of the integers, $\mathbb{Z}$, as an $L_Z$-structure with the standard interpretations.

   (a) Prove: all substructures of $\mathbb{Z}$ are elementarily equivalent to $\mathbb{Z}$, and there are infinitely many.

   (b) Prove: no proper substructure (in fact, submodel) of $\mathbb{Z}$ is elementary.

   (c) Prove or disprove: therefore the complete theory of $\mathbb{Z}$ has quantifier elimination.

   (Part One continues on the next page.)
6. Suppose $L_E$ is a language whose single non-logical symbol is a binary relation symbol $E$. Consider the $L_E$-theory $T$ of all $L_E$-structures in which $E$ defines an equivalence relation with two infinite $E$-classes.

(a) Write down an $L_E$-axiomatization of $T$.

(b) How many non-isomorphic countable models does $T$ have?

(c) How many non-isomorphic models does $T$ have of cardinality $\aleph_1$?

(d) Conclude what you can about the completeness of $T$. 
Part Two (42 points)

Do THREE of the following six problems.

1. Show (in ZF): for a limit ordinal $\lambda$, the cofinality of $\lambda$ is the unique regular cardinal $\kappa$ such that there is an increasing function from $\kappa$ to $\lambda$ whose range is unbounded in $\lambda$.

2. Show (in ZF): if $\lambda$ is a singular ordinal, then there are a stationary subset $S$ of $\lambda$ and a function $f : S \to \lambda$ such that for every $\alpha \in S$, $f(\alpha) < \alpha$, and such that $f$ is not constant on any stationary subset of $S$.

   Is this true if $\lambda$ is regular? Why or why not?

3. Suppose $A \subseteq \omega$ is recursively enumerable, and for every $e \in A$, $W_e = \omega$, that is, $\varphi_e$ is total. Show that there is an $x \in \omega$ such that $W_x = \omega$ and $\varphi_x \neq \varphi_e$ for every $e \in A$.

4. Suppose ZFC is consistent. Show that there is a Turing program $\varphi_e$ that never terminates, on any input, but such that ZFC does not prove that it never terminates.

5. Let $T$ be a deductively closed, consistent and decidable set of sentences (say in a first order language with finitely many symbols). Show that there is a complete, consistent, decidable $T' \supseteq T$.

6. For a model $M$ of Peano Arithmetic, let $D(M)$ be the set of $a \in M$ such that $a$ is definable in $M$ without parameters (that is, for some formula $\varphi(x)$, $M \models \varphi(a)$, but $M \models \neg \varphi(b)$ for all $b \in M$ with $b \neq a$). As is common, we will say that a set $A \subseteq M$ is definable in $M$ if there is a formula $\varphi(x)$ such that $A = \{a \mid M \models \varphi(a)\}$. It is said to be definable in $M$ using parameters if for some $\bar{b} \in M$ and some formula $\varphi(x, \bar{y})$, $A = \{a \mid M \models \varphi(a, \bar{b})\}$.

   (a) Show that there is a (countable) model $M$ of Peano Arithmetic such that $D(M)$ is not definable in $M$ (even allowing the use of parameters). In fact, if the standard model $N$ is a proper elementary submodel of $M$, then $M$ is as wished.

   (b) Show that every model $N$ of Peano Arithmetic has a countable elementary submodel $M$ such that $D(M)$ is definable in $M$. 