The Department of Mathematics
The CUNY Graduate Center
Real Analysis Qualifying Exam
September 2015

Your name: ______________________________________

Do any 8 of the following 12 problems, and put a check below next to each of the problems you want graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______
7. ______
8. ______
9. ______
10. ______
11. ______
12. ______

Further Instructions/Information

- *Use only one side of each sheet.* Attach extra sheets if necessary, use only one side of each of those sheets, and make sure your name is on each of those sheets.
- You have three hours.
- Lebesgue measure is always denoted m.

Do any 8 of the following 12 problems, and please be clear about those you want graded. Lebesgue measure is always denoted m.

1. Show that a closed interval on the real line is not the union of countably many disjoint, closed non-empty sets.

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $f_n \in L^1(\mu)$, and $\|f_n\|_1 \leq n^{-3/2}$ for $n = 1, 2, 3, \ldots$. Show that $f_n(x) \to 0$ for μ-almost every $x \in X$.

3. Let $p \geq 1$, and suppose $f \in L^p(\mathbb{R})$ with respect to Lebesgue measure. Prove that

$$\lim_{x \to \infty} \int_x^{x+1} f \, dm = 0.$$

4. Let f be a continuous real-valued function defined on some closed interval $[a, b] \subset \mathbb{R}$, and suppose $|f|$ is absolutely continuous. Prove that $f : [a, b] \to \mathbb{R}$ is absolutely continuous.

5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that for every $c \in \mathbb{R}$ the equation $f(x) = c$ has at most two solutions in $[0,1]$. Show that the derivative f' exists almost everywhere in $[0,1]$.

6. Let $(\mathbb{R}, \mathcal{M}, m)$ denote the usual Lebesgue measure space, and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and has the property that $g^{-1}(N) \in \mathcal{M}$ whenever $N \in \mathcal{M}$ has measure zero. Prove that $f \circ g$ is measurable, and give an example to show this is not necessarily true if g is only assumed continuous.

7. Let μ be a regular Borel measure on $\mathbb{R}$, and assume there is a non-negative constant C such that

$$\mu(I) \leq C \cdot m(I)$$

for all open intervals $I \subset \mathbb{R}$. Show that μ is absolutely continuous with respect to m, and that the Radon-Nikodym derivative $f = d\mu/dm$ is in $L^\infty(m)$.

8. Let μ be the counting measure on $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and let $X = Y = \mathbb{Z}^+$. Define $f : X \times Y \to [0, \infty)$ by

$$f(x,y) := \begin{cases} 2 - 2^{-x} & \text{if } \; x = y \\ -2 + 2^{-x} & \text{if } \; x = y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the two iterated integrals of f exist but are not equal. What missing hypothesis is responsible for the failure of Fubini's Theorem?

9. Let $f : [0,1] \to \mathbb{R}$ be measurable with the property that $f > 0$ almost everywhere. Suppose $\{E_n\}$ is a sequence of measurable subsets of $[0,1]$ such that $\int_{E_n} f\,dm \to 0$ as $n \to \infty$. Show that $m(E_n) \to 0$ as $n \to \infty$.

10. Prove that for any x, y in any normed linear space one has

$$\left| \|x\|^2 - \|y\|^2 \right| \leq \|x - y\| \|x + y\|.$$

11. (Do Both Parts)

(a) State Minkowski's inequality for L^p spaces.

(b) Assuming $1 < p < \infty$ prove that if $\|f\|_p = \|g\|_p = 1$, $f \neq g$ and $h := \frac{1}{2}(f+g)$, then $\|h\|_p < 1$.

12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that there are constants A and B such that $|f(x)| \leq A|x| + B$ for every $x \in \mathbb{R}$.