NAME :

THE DEPARTMENT OF MATHEMATICS
THE CUNY GRADUATE CENTER
REAL ANALYSIS QUALIFYING EXAM

Do any SEVEN of the following ten problems, and put a check below next to each of the problems you want to be graded.

(1) ____
(2) ____
(3) ____
(4) ____
(5) ____
(6) ____
(7) ____
(8) ____
(9) ____
(10) ____

- **You have three hours.**
- **Use only one side of each sheet. Attach extra sheets if needed.**
- **Print your name clearly on EACH sheet.**

Date: August 20, 2018.

(1) Prove that the product of two absolutely continuous functions $f, g : [0, 1] \to \mathbb{R}$ is absolutely continuous.

(2) Let (X, d) be a compact metric space, and let $T : X \to X$ be a map such that $d(T(u), T(v)) < d(u, v)$, $\forall u, v \in X$. Show that T has a unique fixed point.

(3) Let $I = [a, b] \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function. Let D be a countable subset of I such that f is differentiable on $I \backslash D$. Assume there is a constant M such that $f'(x) \leq M$ for all $x \in I \backslash D$. Prove that

$$f(b) - f(a) \leq M(b - a).$$

(4) Let $p \in [1, \infty)$ and $X = (X, \mu)$ be a measure space. If $f \in L^p(X)$, prove that

$$\mu\{x \in X : |f(x)| \geq t\} \leq \left(\frac{\|f\|_p}{t}\right)^p,$$

for $t \in (0, \infty)$.

(5) Let X be a Banach space with a countably infinite dimension. Then there is no norm on X with respect to which X is complete.

(6) Let X be a metric space and let

$$C_b(X) = \{f|f : X \to \mathbb{R} \text{ is continuous and bounded}\}.$$

We denote $\|f\| = sup_{x\in X}|f(x)|$. Prove that $(C_b(X), \|\cdot\|)$ is a Banach space.

(7) Let X be a compact metric space and $\{f_n\}_{n\geq 1}$ be a sequence which converges in $L^p(X)$ for every $p \in [1, \infty)$, is it true that this sequence must converge a.e.? Give a proof if you claim this is true, otherwise give a counterexample.

(8) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Prove that its derivative f' (which may not be continuous) is Borel measurable.

(9) Let (X, μ) be a measure space and $p \in (0,1)$. If functions $f, g \in L^p(X)$, then prove their sum $f + g \in L^p(X)$, and

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p.$$

(10) Suppose $f \in L^1(\mathbb{R})$. Prove that there is a Borel function $g : \mathbb{R} \to \mathbb{C}$ such that $f = g$ a.e.