Name (Print clearly): ______________________________

Real Variables Qualifying Exam
The Graduate Center, CUNY, August 2019

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
TOTAL	60	

Instructions:

(1) This exam contains nine problems, but at most six problems will be graded. Please clearly list which problems you wish to be graded below:

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(2) Use only one side of each sheet. Do at most one problem on each page.

(3) Write your name on each page. If you include additional pages, write your name on those as well.

(4) Justify your answers. Where appropriate, state without proof the results you are using.

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Problem 1.

Construct an open set $A \subset [0,1]$ such that it is dense in $[0,1]$ with the Lebesgue measure $\mu(A) < 1$, and that $\mu(A \cap (a,b)) > 0$ for any interval $(a,b) \subset [0,1]$.

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Problem 2.

State and prove Lebesgue's dominated convergence theorem.

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Problem 3.

Let $f : \mathbb{R} \to \mathbb{R}$. Assume that for every $c \in \mathbb{R}$ the level set $\{x \in \mathbb{R} : f(x) = c\}$ is Lebesgue measurable. Is it true that f is Lebesgue measurable? You must justify your answer.

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Problem 4.

Let $f(x)$ be Lebesgue integrable on $(0,1)$. For $x \in (0,1)$ we define $g(x) = \int_x^1 \frac{f(t)}{t} dt$. Prove that g is Lebesgue integrable on $(0,1)$, and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

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Problem 5.

Let X be a Banach space and X^* be its dual.
(a) Prove that if X^* is separable then so is X.

(b) Use (a) to show that ℓ^1 is not reflexive.

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Problem 6. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin(1/x^2), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Is f of bounded variation? You must justify your answer.

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Problem 7.

Let $E \subset \mathbb{R}^n$ be a set of positive measure. Suppose that functions $f, g \in L^3(E)$, and that they satisfy

$$\|f\|_3 = \|g\|_3 = \int_E f^2(x) g(x) dx = 1.$$

Prove that $g(x) = |f(x)|$ for almost every $x \in E$.

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Problem 8.

Prove that if f is measurable on $[0,1]$ and $f(x) \geq 1$, then

$$\int_0^1 f(x)\ln[f(x)]dx \geq \left(\int_0^1 f(x)dx\right)\left(\int_0^1 \ln[f(x)]dx\right).$$

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Problem 9.

Let f be a bounded linear functional on some subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^{\perp}$.