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**Real Variables Qualifying Exam**  
**The Graduate Center, CUNY, December 2020**

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
TOTAL	50	

**Instructions:**

- (1) The exam will be conducted while on Zoom. Have a camera turned on so that you are visible to the proctor for the entire exam.
- (2) The exam will be 2 hours and 30 minutes followed by a short discussion part having to do with the questions that were asked on the exam (held in a breakout room). When you complete the exam, scan the pages you want graded to PDF (camera app is fine) and send it to your proctor. The discussion will occur after the proctor receives your scanned answers. The intent of the discussion is to allow you to address incomplete answers.
- (3) This exam contains eight problems, but at most five problems will be graded. Please clearly list these here or on the first page scanned:  
  
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- (4) If possible, print out this exam and work on individual problems on the printed sheets. Use of blank white paper is also acceptable. Do at most one problem on each page, and be sure to write your name on each.
- (5) Justify your answers. Where appropriate, state without proof the results you are using. Each part of a problem counts equally.

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**Problem 1.**

Let  $X$  be a locally compact and  $\sigma$ -compact metric space and  $x_0 \in X$ . For each  $x \in X$  define the function

$$B_x : X \rightarrow \mathbb{R}; \quad B_x(y) = d(x, y) - d(x, x_0).$$

Equip the space  $\mathcal{C}(X)$  of continuous functions  $X \rightarrow \mathbb{R}$  with the topology of uniform convergence on compact sets. Prove that any sequence in  $\{B_x : x \in X\}$  has a subsequence that converges in  $\mathcal{C}(X)$ .

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**Problem 2.**

Let  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous map, and suppose there exists a constant  $M > 0$  such that  $\|\rho(x) - x\| \leq M$ , for any  $x \in \mathbb{R}^n$ . Prove that the map  $\rho$  is surjective.

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**Problem 3.**

Consider  $L^\infty = L^\infty([0, 1], \mathcal{B}, m)$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[0, 1]$  and  $m$  is Lebesgue measure.

- (1) Prove that the evaluation map  $f \mapsto f(0)$  is a bounded linear function on  $\mathcal{C}([0, 1])$ .
- (2) Use the Hahn-Banach theorem, to argue there exists a  $\ell \in (L^\infty)^*$  such that  $\ell(f) = f(0)$  for any  $f \in \mathcal{C}([0, 1]) \subset L^\infty$ .
- (3) Prove that  $\ell$  is not given by the integral over an  $L^1$  function by considering the sequence  $f_n \in \mathcal{C}([0, 1])$

$$f_n(x) = \max \{1 - nx, 0\}.$$

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**Problem 4.**

Let  $m$  denote Lebesgue measure on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  and let  $\mu$  be the measure which is absolutely continuous with respect to  $m$  and has Radon-Nikodym derivative  $\frac{d\mu}{dm}(\theta) = \cos \theta$ . Consider the Hilbert space  $L^2([-\frac{\pi}{2}, \frac{\pi}{2}], \mathcal{B}, \mu)$ . Find an orthogonal basis for the subspace of polynomials of degree less than or equal to two. Show your work.

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**Problem 5.**

Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be measurable, and  $t \in (0, 1)$ . Suppose for any  $x, y > 0$ , we have

$$f(x + y) = tf(x) + (1 - t)f(y),$$

prove that  $f$  is constant.

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**Problem 6.**

Let  $f \in L^p \cap L^\infty(X, \mathcal{M}, \mu)$  for some  $1 \leq p < \infty$ . Prove that  $f \in L^q$  for all  $q > p$  and

$$\lim_{q \rightarrow \infty} \|f\|_q = \|f\|_\infty.$$

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**Problem 7.**

Let  $H$  be a Hilbert space with orthonormal basis  $\{b_\alpha : \alpha \in I\}$  where  $I$  is some indexing set. Let  $\{s_\alpha \geq 0 : \alpha \in I\}$  be a selection of non-negative real numbers such that  $\sum_{\alpha \in I} s_\alpha < \infty$ . Prove that the set

$$X = \{x \in H : |\langle x, b_\alpha \rangle|^2 \leq s_\alpha \text{ for all } \alpha \in I\}$$

is compact.



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**Problem 8.**

Compute the following limits and justify your answer:

(1)

$$\lim_{n \rightarrow \infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin \frac{x}{n} \, dx,$$

(2)

$$\lim_{n \rightarrow \infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx .$$