Qualifying Exam, Real Analysis
August 2021

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
TOTAL	50	

Instructions:

1. The exam will be conducted while on Zoom. Have a camera turned on so that you are visible to the proctor for the entire exam.

2. The exam will be 2 hours and 30 minutes followed by a short discussion part having to do with the questions that were asked on the exam (held in a breakout room). When you complete the exam, scan the pages you want graded to PDF (camera app is fine) and send it to your proctor. The discussion will occur after the proctor receives your scanned answers. The intent of the discussion is to allow you to address incomplete answers.

3. This exam contains eight problems, but at most five problems will be graded. Please clearly list these here or on the first page scanned:

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4. If possible, print out this exam and work on individual problems on the printed sheets. Use of blank white paper is also acceptable. Do at most one problem on each page, and be sure to write your name on each.

5. Justify your answers. Where appropriate, state without proof the results you are using. Each part of a problem counts equally.

Problem 1
Let $(X, \mathcal{F})$ be a measurable space, μ be a measure on $(X, \mathcal{F})$ and $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$.

(a) Show that
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

(b) Show that if $A_1 \supset A_2 \supset \dots$ and $\mu(A_1) < \infty$ then
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(c) Show that
$$\sum_{n=1}^{\infty} \mu(A_n) < \infty \quad \Rightarrow \quad \mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = 0.$$

Does the converse statement hold true? Justify your answer.

Problem 2
Let $(X, \mathcal{F}, \mu)$ be a measure space and $(f_n)_{n\geq 1}$ be a sequence of non-negative measurable functions on $(X, \mathcal{F})$.

(a) Derive Fatou's lemma from the monotone convergence theorem.

(b) Prove that if $f_n \to f$ in measure then
$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Problem 3
Give an example of a bounded Lebesgue measurable function f on $[0, 1]$ such that every function which is equal to f a.e. on $[0, 1]$ is not Riemann integrable.

Problem 4
Let $a > 1$ be a real number, and for each real sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ define the new sequence
$$T_a(\alpha) := \left(\frac{\alpha_n}{a^n}\right)_{n\in\mathbb{N}}$$
where $\mathbb{N} := \{1, 2, \cdots\}$.

(a) Show that T_a defines a linear continuous map from ℓ^p to ℓ^1 for each $1 \leq p \leq \infty$.

(b) Show that for each $1 \leq p \leq \infty$ the operator norm of T_a is given by
$$\sup_{\xi\in\ell^p, \|\xi\|\leq 1} \|T_a \xi\| = \begin{cases} \dfrac{1}{a} & \text{if } p = 1 \\ \left(\dfrac{1}{a^{p'} - 1}\right)^{1/p'} & \text{if } 1 < p \leq \infty \end{cases} \qquad (\frac{1}{p} + \frac{1}{p'} = 1).$$

Problem 5

Let $\mathcal{X}$ be a Banach space. We consider the set $\mathcal{L}(\mathcal{X},\mathcal{X})$ of bounded linear operators from $\mathcal{X}$ to itself. Prove that if $|\lambda| > \|T\|$ then $\lambda I - T$ is bijective and $(\lambda I - T)^{-1}$ is continuous.

Hint: Consider $\frac{1}{\lambda}\sum_{n=0}^{\infty}(T/\lambda)^n$ *as a candidate for* $(\lambda I - T)^{-1}$.

Problem 6

Given a non-empty Hausdorff compact topological space K, consider the space of continuous functions $E := C(K)$ endowed with the usual sup-norm. For each $x \in K$ define the map:

$$T_x : C(K) \to \mathbb{R}, \qquad \xi \mapsto \xi(x).$$

(a) Show that $\tau := \{T_x : x \in K\}$ is a bounded subset of the dual $(E^*, \|\cdot\|_*)$ normed space.

(b) Under the additional condition that K is a metric space of infinite cardinality, show that the set τ is never compact.

(c) If K is a metric space, show that that the map

$$K \to E^*, \qquad x \mapsto T_x$$

is injective.

Problem 7

Let $f \in L^p(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), m)$. Let τ_y be the shifting operator $\tau_y f(x) = f(x-y)$. Show that for $1 \leq p < \infty$ we have $\lim_{y\to 0}\|\tau_y f - f\|_p = 0$.

Problem 8

(a) Consider a map $F : X \to Y$ between two metric spaces, and let $a \in X$. Assume that for each sequence $x_n \to a$ there exists a subsequence x_{n_k} such that $F(x_{n_k}) \to F(a)$. Prove that F is continuous at a.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^1(\mathbb{R})$ such that $f, f' \in L^\infty(\mathbb{R})$. Consider also the measure space $([0,1], m)$. For a given $u \in L^2([0,1])$, we define the following functional on $L^2([0,1])$

$$\begin{aligned} \ell_u &: L^2([0,1]) \to \mathbb{R} \\ &\xi \mapsto \ell_u(\xi) := \frac{d}{dt}\Big[\int_{[0,1]} f\big(u(x) + t\xi(x)\big)dx\Big]\Big|_{t=0}. \end{aligned}$$

Show that we have for any $\xi \in L^2([0,1])$

$$\ell_u(\xi) = \int_{[0,1]} \xi(x) f'(u(x))dx.$$

(c) By considering the dual $[L^2([0,1])]^*$ with the usual norm, prove that the map

$$\begin{aligned} L^2([0,1]) &\to [L^2([0,1])]^* \\ u &\mapsto \ell_u \ . \end{aligned}$$

is continuous.
Hint: (a) is useful.