NAME :

THE DEPARTMENT OF MATHEMATICS
THE CUNY GRADUATE CENTER
REAL ANALYSIS QUALIFYING EXAM

Do any EIGHT of the following twelve problems, and put a check below next to each of the problems you want to be graded.

(1) ____
(2) ____
(3) ____
(4) ____
(5) ____
(6) ____
(7) ____
(8) ____
(9) ____
(10) ____
(11) ____
(12) ____

- **You have three hours.**
- **Use only one side of each sheet. Attach extra sheets if needed.**
- m **always denotes the Lebesgue measure.**

Date: May 23 2016.

(1) Is it possible to approximate every continuous function on $[0, \infty)$ uniformly (with arbitrary precision) by polynomials? Prove if it is possible or give a counterexample otherwise.

(2) Let $E_1, E_2, \ldots,$ be Lebesgue measurable subsets of $\mathbb{R}$ with $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, where μ denotes Lebesgue measure on $\mathbb{R}$. Let

$$A = \{x \in \mathbb{R} \colon x \in E_k \text{ for infinitely many } k \in \mathbb{N}\}.$$

Prove that A is Lebesgue measurable and $\mu(A) = 0$.

(3) Prove that the set of irrational numbers cannot be written as a countable union of closed sets in the topology of $\mathbb{R}$.

(4) Prove that the space of non-invertible linear operators on $\mathbb{R}^n$ is closed and connected.

(5) Prove Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and then use that inequality to prove Hölder's Inequality. Here $1 < p < \infty$, $1/p + 1/q = 1$, and $a, b \in (0, \infty)$.

(6) Let V be a Hilbert space and let $T : V \to V$ be a linear operator satisfying $\|T\| \leq 1$. Prove that $T - \sqrt{2}I$ is invertible, or give a counterexample.

(7) Define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \cos(1/x^2) & \text{otherwise.} \end{cases}$$

Prove that f is differentiable at all points $x \in (-1, 1)$, but is not of bounded variation.

(8) Let X be a non-empty set, and for $j = 1, 2$ let $\mathcal{M}_j$ be a σ-algebra on X with associated positive measure μ_j. Prove that μ_1 and μ_2 generate the same outer measure on X if and only if $\mu_i = \mu_j^*$, on $\mathcal{M}_i$ for $i \neq j$, where μ_j^* stands for the outer measure on X defined by μ_j.

(9) Prove that a function $f : \mathbb{R} \to \mathbb{R}$ that is monotone is Borel measurable.

(10) Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Prove that

(a) Any sequence $(e_n)_{n\geq 1}$ of orthonormal elements converges weakly to 0.

(b) Prove that any x in the unit ball $\{x : \|x\| \leq 1\}$ is a weak limit of a sequence in the unit sphere $\{x : \|x\| = 1\}$, i.e., the unit sphere is dense in the unit ball in the weak topology.

(11) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $1 \leq p < \infty$. Prove that if a sequence of functions f_n converges to f in L^p, then f_n converges to f in measure. In the other direction, if f_n converges to f in measure and $|f_n| \leq g$ for some function $g \in L^p$, then $f_n \to f$ in L^p.

(12) Consider $f \in L^1(X, \mathcal{M}, \mu)$. We define the measure

$$\nu(E) = \int_E f \, \mathrm{d}\mu, \; E \in \mathcal{M}.$$

Describe the Hahn decomposition of ν. What is the total variation of ν in terms of f and μ?