Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Fall 2016

Instructions: Do **8 problems** in total, with **exactly two problems** from Part I, and **at least two problems** from each of Parts II and III. Justify your answers and include the names or the precise statements of theorems that you cite.

Part I

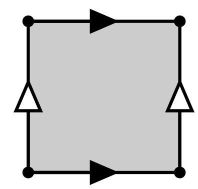
1. A map $f : X \to Y$ is *proper* if and only if the preimage of compact sets is compact. Prove that a space X is Hausdorff if and only if the diagonal map $X \to X \times X$ is proper.

2. Prove that the product of an arbitrary collection of connected spaces is connected.

3. Prove that the following two descriptions of the torus are homeomorphic:

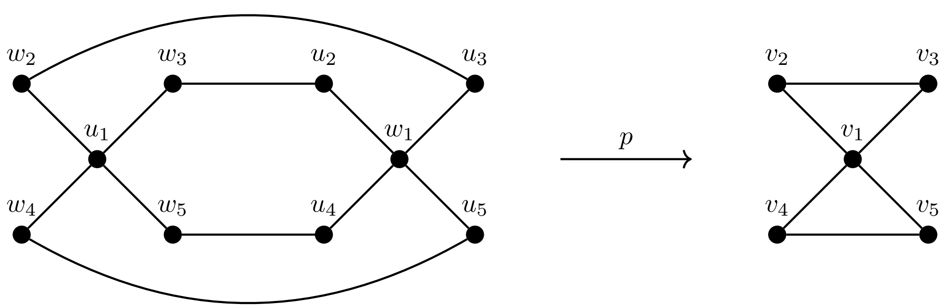
   (a) $S^1 \times S^1$

   (b) A square with opposite sides identified as shown.



Part II

4. The map $p : C \to G$ defined by $\overline{u_i w_j} \mapsto \overline{v_i v_j}$ defines a covering map from the graph C on the left to the graph G on the right.



   Describe the automorphism group $\mathrm{aut}(p)$ of this cover and the image of the fundamental group under the map $p_* : \pi_1(C, u_1) \to \pi_1(G, v_1)$.

5. A topological group is a topological space G together with continuous product and inverse maps, $G \times G \to G$ and $G \to G$ that make the G into a group. Prove that if G is a topological group and $e \in G$ is the identity, then $\pi_1(G, e)$ is abelian.

6. Let $X = S^1 \cup_f D^2$ where $f(z) = z^5$.

   (a) Compute $\pi_1(X)$.

   (b) Prove that any map $g : X \to S^1$ is null-homotopic.

7. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus.

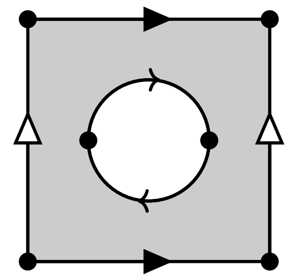
   (a) Let L be a line of non-zero rational slope p/q in $\mathbb{R}^2$. Prove that L projects to a homotopically non-trivial curve $\alpha_{p,q}$ in T.

   (b) Let $X = T \cup_f D^2$, where $f : \partial D^2 \to \alpha_{p,q}$ identifies S^1 to $\alpha_{p,q}$. Compute $\pi_1(X)$.

8. Let X be the surface as shown.

   (a) Compute $\pi_1(X)$.

   (b) Identify the surface using the classification of surfaces.



Part III

9. The suspension ΣX of X is the quotient space

$$\Sigma X = (X \times [0,1])/\{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}.$$

   (a) Prove that $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$.

   (b) Compare $H_1(\Sigma X)$ and $H_0(X)$.

10. Let X be the subset of $\mathbb{R}^3$ consisting of the union of the spheres $A = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : \{(x-1)^2 + y^2 + z^2 = 1\}$. Compute $H_*(X)$.

11. Let $T^n = (S^1)^n$ be the n-torus. Compute $\chi(T^n)$.

12. Let T be the torus $S^1 \times S^1$, and let X be the one point union of two copies of S^1 and S^2, i.e. $S^1 \vee S^1 \vee S^2$. Show that T and X have the same homology groups. Describe the cup product ring structure in each case and deduce that they are not homotopy equivalent.

13. Define the Lefschetz number Λ_f of a map $f : X \to X$ to be

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \operatorname{trace} \left(H_k(X, \mathbb{Q}) \xrightarrow{f_*} H_k(X, \mathbb{Q}) \right).$$

Compute the Lefschetz number of the map $f : T^2 \to T^2$ obtained by flipping the 2-dimensional torus upside down.